

1

Liebmann technical documentation



2

3

Laplace equation 2D (ZR)

4

(Cylindrical coordinates).

5

relaxation scheme explained.

6

(5 - point star)

7

author: Marcin Kulbaka

8

email: mkulbaka@onet.pl

9

project homepage: http://marcinkulbaka.prv.pl/Liebmann/index_en.html

10

license: GNU General Public License v.3.0+

11

version 13

12

2024.12.13

13

University of Maria Curie - Skłodowska in Lublin, Poland

14	Contents	
15	1 Liebmann technical documentation series	4
16	2 Versions of this document	4
17	3 Solving Laplace equation using relaxation method	5
18	4 Explanation of symbols in calculations	6
19	5 Mesh ZR - type A (on axis)	7
20	6 Mesh ZR - type B (on axis)	8
21	7 Mesh ZR - type C (on axis)	9
22	8 Mesh ZR - type D (on axis)	10
23	9 Example of A-type mesh in ANSI C (on axis)	11
24	10 Example of B-type mesh in ANSI C (on axis)	14
25	11 Example of C-type mesh in ANSI C (on axis)	15
26	12 Example of D-type mesh in ANSI C (on axis)	17
27	13 Partial derivatives on Oz axis	18
28	13.1 Personal note	18
29	13.2 Nodes numbering (on axis Oz)	18
30	13.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates . . .	18
31	13.4 Laplace operator in rotationally symmetrical systems	18
32	13.5 Value of first partial derivative of V with respect to r on axis Oz .	19
33	13.6 Value of second partial derivative of V with respect to r on axis Oz	19
34	13.7 Value of first partial derivative of V with respect to r divided by r	
35	on axis Oz	20
36	13.8 Value of second partial derivative of V with respect to z on axis Oz	21
37	14 Partial derivatives off Oz axis	22
38	14.1 Personal note	22
39	14.2 Nodes numbering in Liebmann mesh (off axis Oz)	22
40	14.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates . . .	22
41	14.4 Laplace operator in rotationally symmetrical systems	23
42	14.5 Value of second partial derivative of V with respect to r off axis Oz	23
43	14.6 Value of first partial derivative of V with respect to r divided by r	
44	off axis Oz	23
45	14.7 Value of second partial derivative of V with respect to z off axis Oz	23

46	15 Relaxation formula for node P1 (on axis Oz)	24
47	15.1 Node description	24
48	15.2 Calculation of relaxation formula	24
49	15.3 Final forms of relaxation formula	25
50	15.3.1 zrLV_RELAX5_P1_ON_A	25
51	15.3.2 zrLV_RELAX5_P1_ON_B	25
52	15.3.3 zrLV_RELAX5_P1_ON_C	25
53	15.3.4 zrLV_RELAX5_P1_ON_D	25
54	16 Relaxation formula for node P2 (on axis Oz)	26
55	16.1 Node description	26
56	16.2 Calculation of relaxation formula	26
57	16.3 Final forms of relaxation formula	27
58	16.3.1 zrLV_RELAX5_P2_ON_A	27
59	16.3.2 zrLV_RELAX5_P2_ON_B	27
60	16.3.3 zrLV_RELAX5_P2_ON_C	27
61	16.3.4 zrLV_RELAX5_P2_ON_D	27
62	17 Relaxation formula for node P3 (on axis Oz)	28
63	17.1 Node description	28
64	17.2 Calculation of relaxation formula	28
65	17.3 Final forms of relaxation formula	29
66	17.3.1 zrLV_RELAX5_P3_ON_A	29
67	17.3.2 zrLV_RELAX5_P3_ON_B	29
68	17.3.3 zrLV_RELAX5_P3_ON_C	29
69	17.3.4 zrLV_RELAX5_P3_ON_D	29
70	18 Relaxation formula for node P4	30
71	18.1 Node description	30
72	18.2 Calculation of relaxation formula	30
73	18.3 Final forms of relaxation formula	31
74	18.3.1 zrLV_RELAX5_P4_A	31
75	18.3.2 zrLV_RELAX5_P4_B	31
76	18.3.3 zrLV_RELAX5_P4_C	31
77	18.3.4 zrLV_RELAX5_P4_D	31
78	19 Relaxation formula for node P5	32
79	19.1 Node description	32
80	19.2 Calculation of relaxation formula	32
81	19.3 Final forms of relaxation formula	33
82	19.3.1 zrLV_RELAX5_P5_A	33
83	19.3.2 zrLV_RELAX5_P5_B	33
84	19.3.3 zrLV_RELAX5_P5_C	33
85	19.3.4 zrLV_RELAX5_P5_D	33

86	20 Relaxation formula for node P6	34
87	20.1 Node description	34
88	20.2 Calculation of relaxation formula	34
89	20.3 Final forms of relaxation formula	35
90	20.3.1 zrLV_RELAX5_P6_A	35
91	20.3.2 zrLV_RELAX5_P6_B	35
92	20.3.3 zrLV_RELAX5_P6_C	35
93	20.3.4 zrLV_RELAX5_P6_D	35
94	21 Relaxation formula for node P7	36
95	21.1 Node description	36
96	21.2 Calculation of relaxation formula	36
97	21.3 Final forms of relaxation formula	37
98	21.3.1 zrLV_RELAX5_P7_A	37
99	21.3.2 zrLV_RELAX5_P7_B	37
100	21.3.3 zrLV_RELAX5_P7_C	38
101	21.3.4 zrLV_RELAX5_P7_D	38
102	22 Relaxation formula for node P8	39
103	22.1 Node description	39
104	22.2 Calculation of relaxation formula	39
105	22.3 Final forms of relaxation formula	40
106	22.3.1 zrLV_RELAX5_P8_A	40
107	22.3.2 zrLV_RELAX5_P8_B	40
108	22.3.3 zrLV_RELAX5_P8_C	40
109	22.3.4 zrLV_RELAX5_P8_D	41
110	23 Relaxation formula for node P9	42
111	23.1 Node description	42
112	23.2 Calculation of relaxation formula	42
113	23.3 Final forms of relaxation formula	43
114	23.3.1 zrLV_RELAX5_P9_A	43
115	23.3.2 zrLV_RELAX5_P9_B	43
116	23.3.3 zrLV_RELAX5_P9_C	44
117	23.3.4 zrLV_RELAX5_P9_D	44

118 **1 Liebmann technical documentation series**

- 119 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-
120 sacyjną Liebmann. (Polish version / wersja polska)
- 121 2. Determination of electrostatic field distribution by using Liebmann relax-
122 ation method. (English version / wersja angielska)
- 123 3. Graphics. Mapping voltages to colours. (colormaps)
- 124 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme
125 explained. (5 - point star)
- 126 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme
127 explained. (5 - point star)
- 128 6. Liebmann source code. (ANSI C programming language)

129 **2 Versions of this document**

- 130 1. version 1 - 2023.11.03
- 131 2. version 2 - 2023.01.04
- 132 3. version 3 - 2024.02.02
- 133 4. version 4 - 2024.04.02
- 134 5. version 5 - 2024.05.18
- 135 6. version 6 - 2024.05.23
- 136 7. version 7 - 2024.05.24
- 137 8. version 8 - 2024.06.06 (complete $P_1..P_9$)
- 138 9. version 9 - 2024.06.09
- 139 10. version 10 - 2024.07.17
- 140 11. version 11 - 2024.07.18
- 141 12. version 12 - 2024.09.03
- 142 13. version 13 - 2024.12.13

143 **3 Solving Laplace equation using relaxation method**

144 I tried to solve Laplace equation using mainly information from Pierre Grivet's
145 book (Electron Optics) - [1].

146 There are few editions of this book (1965, 1972). Second edition (1972) con-
147 tains explanation of relaxation method (page 38).

148 More generalized approaches has been drafted by James R. Nagel - [2].
149 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).

150
151 Taylor expansion in cylindrical coordinates has been found on the Internet:
152 [3].

153
154 There are also publications edited by Albert Septier: Focusing of Charged
155 Particles [4] and Applied Charged Particle Optics (part A). [5].

156 I have also found some ideas in publication of D W O Heddle: Electrostatic
157 Lens Systems [6] (especially using PC computers to solve electrostatic prob-
158 lems).

159 I have also found (brief) description of by - hand solving of Laplace equa-
160 tion by Bohdan Paszkowski - [7] (Polish edition). English translation of this book
161 also exists - [8].

162
163 I would like to thank many people, who helped me with this challenge. Espe-
164 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),
165 who enabled me to use SIMION and MATLAB software while writing master's
166 thesis about electron optical systems at University of Maria Curie - Skłodowska
167 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-
168 sion about numerical methods. What is more, my colleague Bartosz in 2012
169 had explained me general problems with software efficiency. So he had also
170 contributed significantly to the idea of Liebmann software (especially using C
171 language).

172 4 Explanation of symbols in calculations

- 173 • P_i - i -th mesh node
- 174 • V_i - value of electrostatic potential at node P_i . Unit - [V]
- 175 • h - mesh step (for example h_x - mesh step in x direction). Unit - [mm]
- 176 • $g_{i+/-}$ - gradient in direction i (for example $g_{1z-} = \frac{V_1 - V_{1z-}}{h_z}$). Unit - $[\frac{V}{mm}]$
- 177 • i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, \dots, \text{size_row}$
- 178 • i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, \dots, \text{size_col}$
- 179 • p - in book: - [1] $r = ph_r$, so for off - axis point we have: $p = (i_{row} - 1)$

180 Symbols in final relaxation formulae

181 zrLV_RELAX5_P1_A

- 182 • zr - coordinates (2D, cylindrical)
- 183 • LV - Laplace equation in vacuum (no dielectrics)
- 184 • RELAX5 - 5- point relaxation method
- 185 • P1 - relaxation scheme for point P1 (in general P1 .. P9)
- 186 • A - mesh type A (in general A .. D)

187 **5 Mesh ZR - type A (on axis)**

188 $h_z \neq h_r$

189 gradient V outside a mesh exists

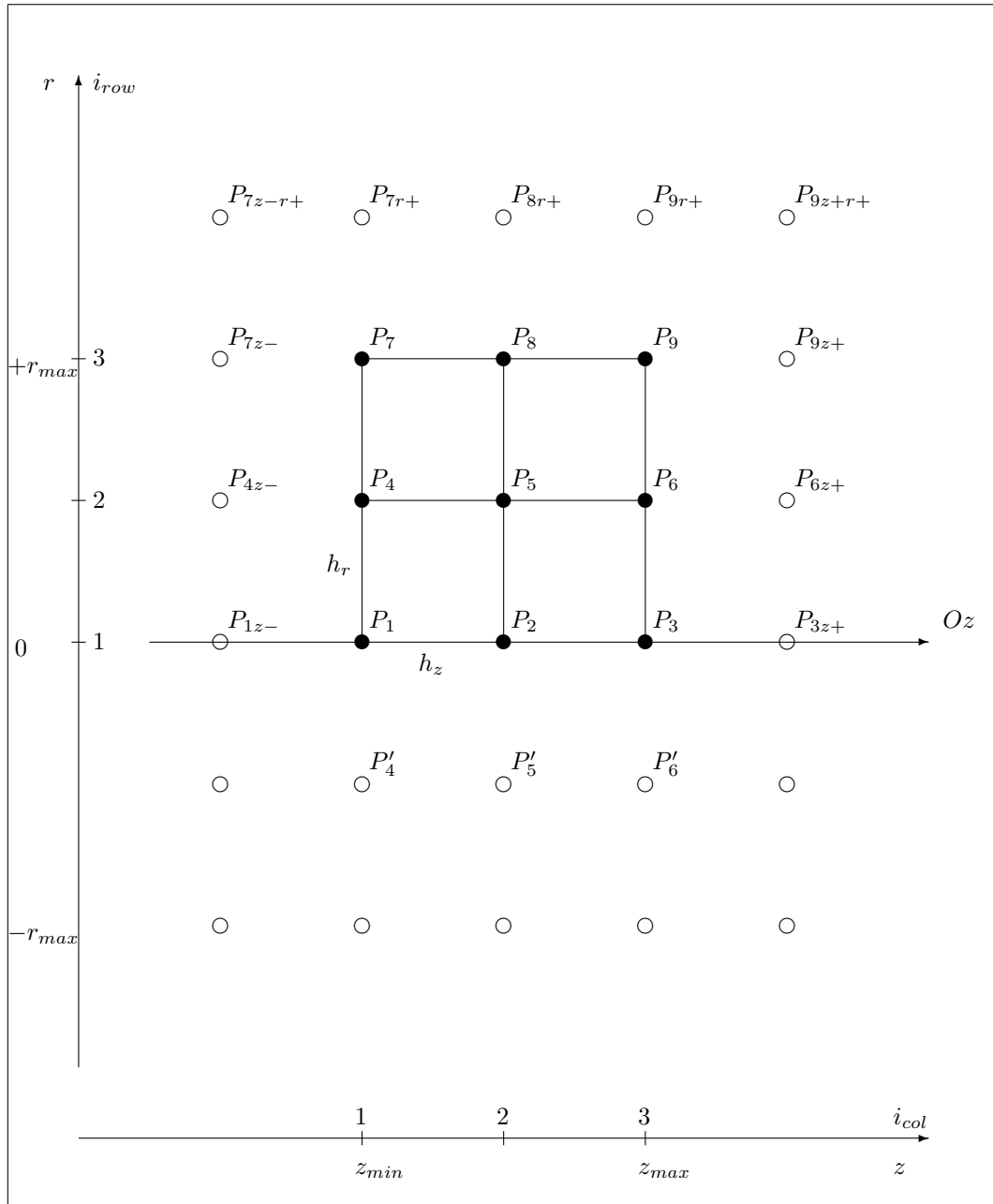


Figure 1: Mesh ZR type A

190 **6 Mesh ZR - type B (on axis)**

191 $h_z \neq h_r$

192 gradient V outside a mesh does not exist

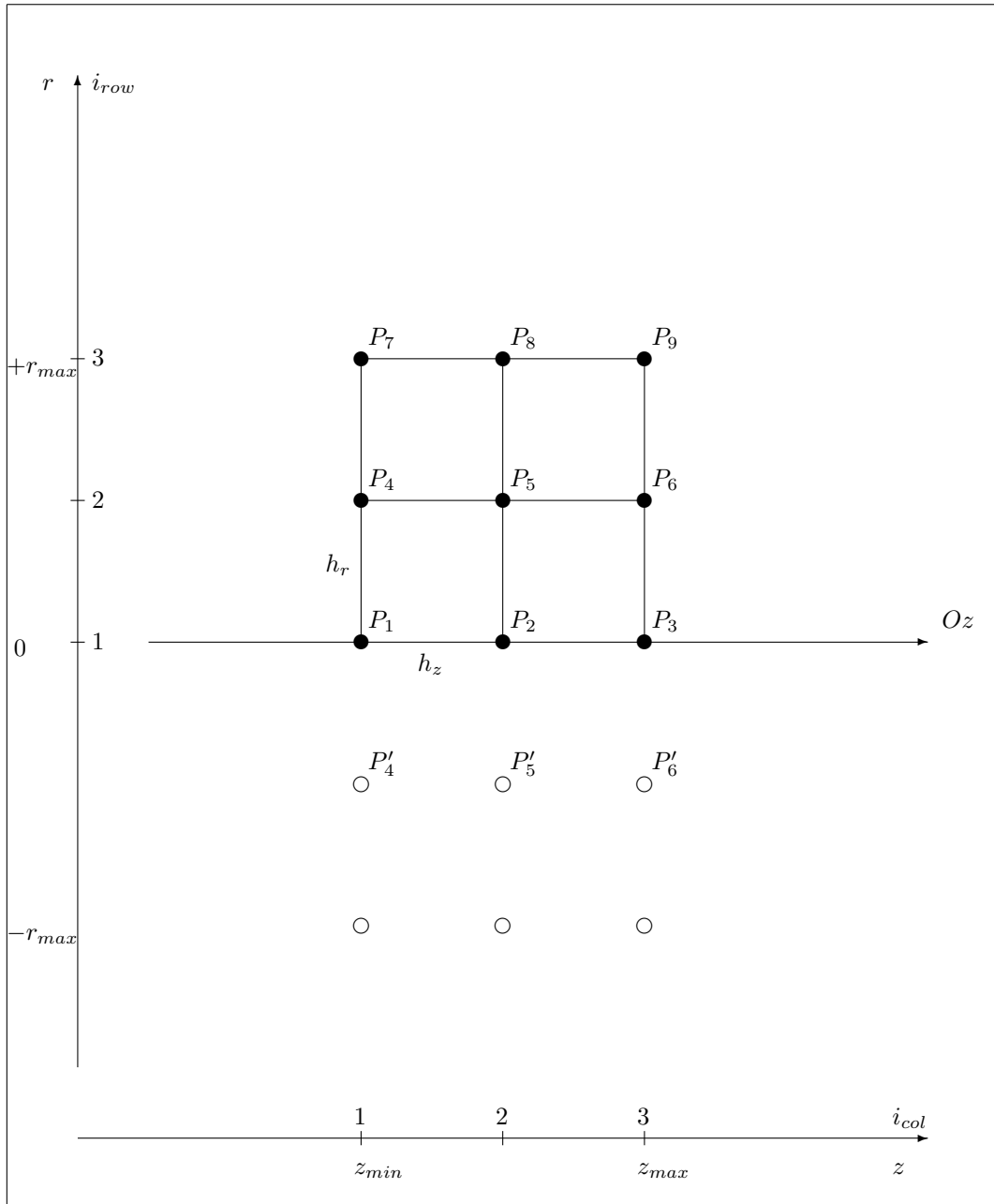


Figure 2: Mesh ZR type B

193 **7 Mesh ZR - type C (on axis)**

194 $h_z = h_r = h$

195 gradient V outside a mesh exists

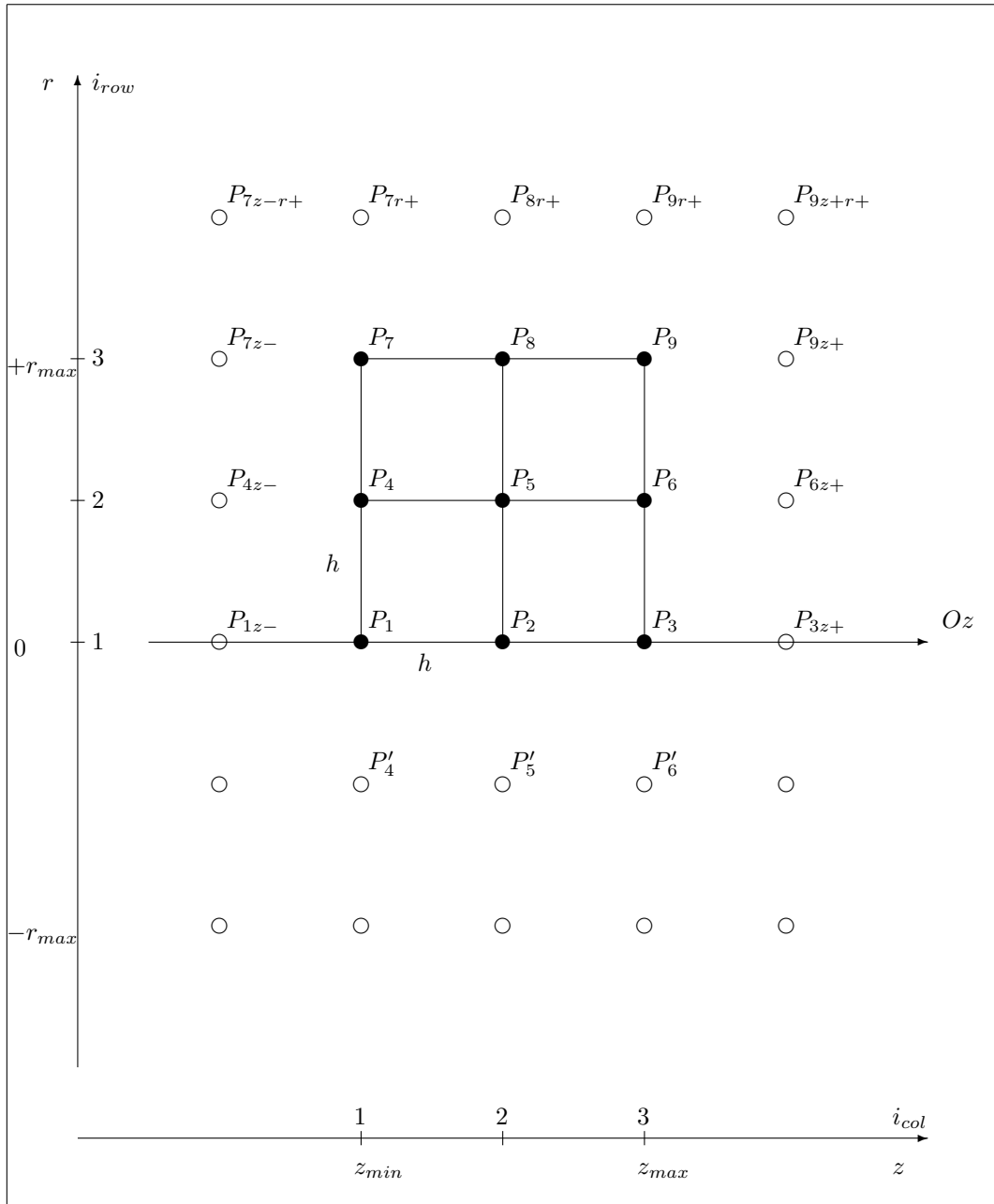


Figure 3: Mesh ZR type C

196 **8 Mesh ZR - type D (on axis)**

197 $h_z = h_r = h$

198 gradient V outside a mesh does not exist

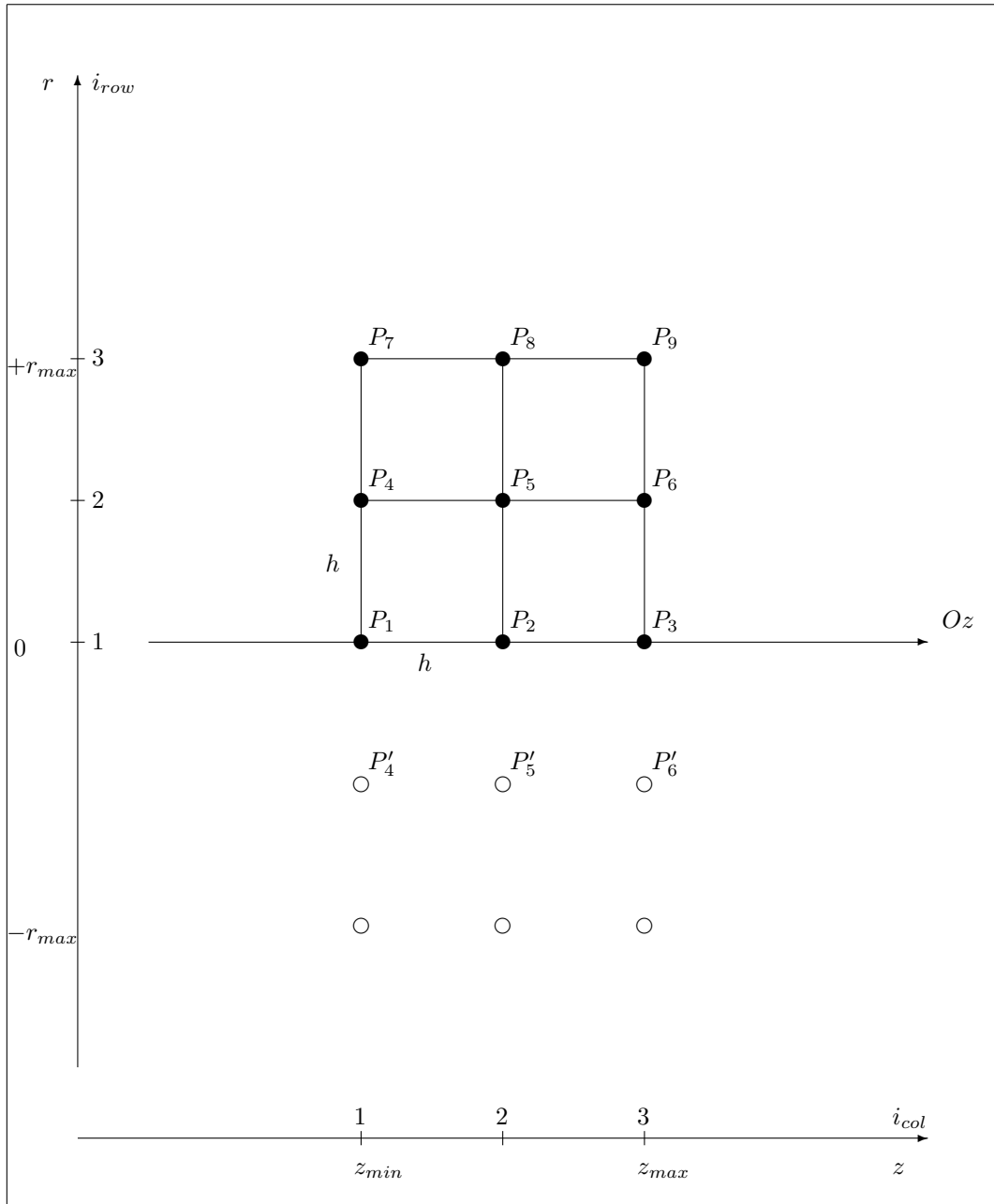


Figure 4: Mesh ZR type D

199 **9 Example of A-type mesh in ANSI C (on axis)**

200 Example of A- type mesh in ANSI C program. The mesh is represented by 2
 201 dimensional array of double precision numbers. Rows and columns in mesh
 202 are numbered from 1 (this was my choice) instead of default 0 (as usual in C
 203 language). This choice has pros and cons. It is easier to calculate mesh size
 204 (size_row * size_col). Access to each node can be also more intuitive, but logic
 205 in each library function must contain this shift between node ordering styles.

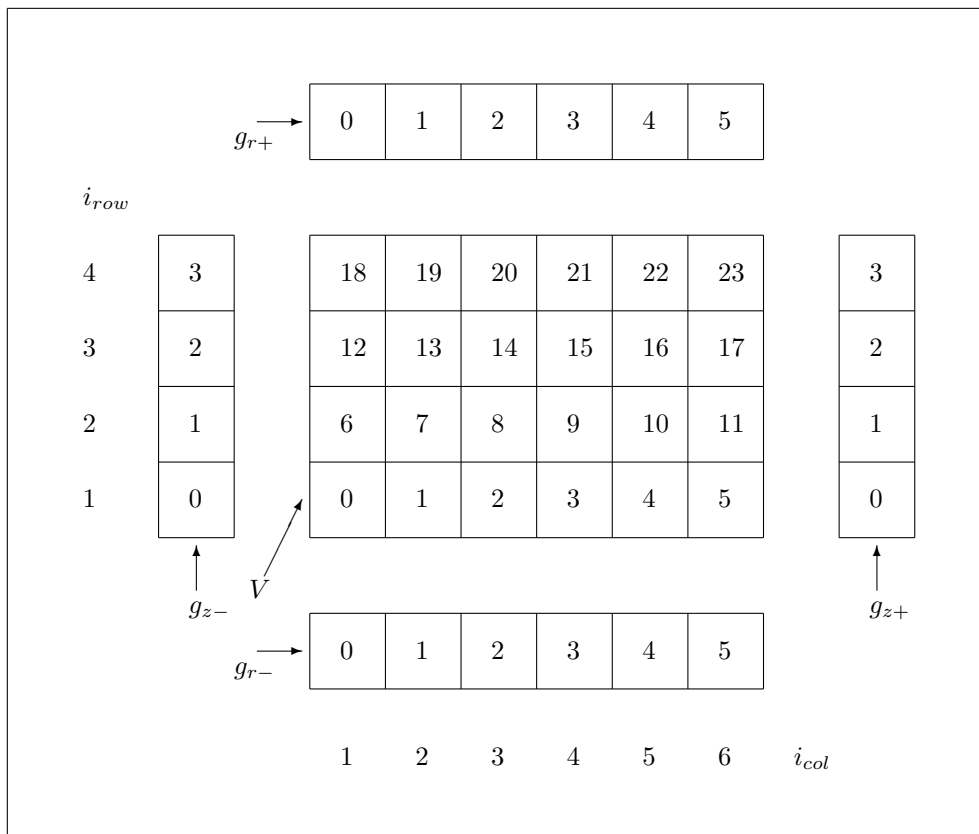


Figure 5: ANSI C - mesh XY type A

206 Note. This is more general example of „off-axis” mesh. If bottom edge of
 207 mesh lies on axis Oz , then gradient g_{r-} does not exist.

- 208 • $g_{z-} \equiv \text{double* ptr_gZ_minus}$
- 209 • $g_{z+} \equiv \text{double* ptr_gZ_plus}$
- 210 • $g_{r-} \equiv \text{double* ptr_gR_minus}$
- 211 • $g_{r+} \equiv \text{double* ptr_gR_plus}$

```

212 •  $V \equiv \text{double}^* \text{ptr}_V$ 
213 • unsigned int size_row == 4
214 • unsigned int size_col == 6
215 • unsigned int i_row == 1, 2, .., 4
216 • unsigned int i_col == 1,2, .., 6
217 • double h_z == 1.0 [mm]
218 • double h_r == 2.0 [mm]

```

219 The following picture describes analogous version of `ptr_V` mesh, which
 220 can be dynamically allocated on heap by pointer metod. The mesh is rep-
 221 resented by single block of memory. The numbers or rows and columns are
 222 also known, so each node can be also accessed by appropriate index (memory
 223 address).

224 The following picture describes analogous version of mesh, which can be
 225 easily dynamically allocated on heap by pointer metod. The mesh is repre-
 226 sented by single block of memory. The numbers or rows and columns are also
 227 known, so each node can be also accessed by appropriate index (memory ad-
 228 dress).

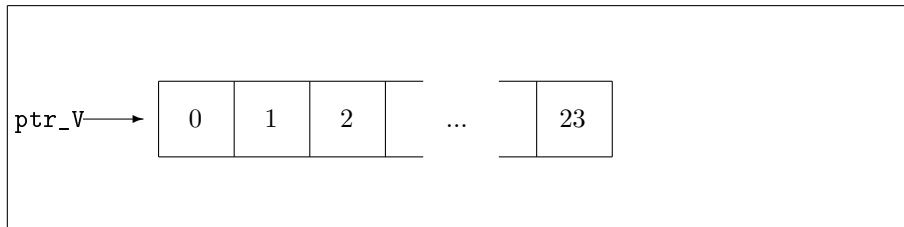


Figure 6: ANSI C - mesh ZR type D

229 Each mesh point has its unique index (let's say `icp` - (index of central
 230 point)), which can be determined, if we know indices of row and column (`i_row`,
 231 `i_col`).

$$\text{icp} == (\text{i_row} - 1) * \text{size_col} + \text{i_col} - 1 \quad (9.1)$$

232 For example for each point of a mesh indices of row and column have val-
 233 ues:

$$\begin{aligned} \text{i_row} &== 1, 2, \dots, \text{size_row} \\ \text{i_col} &== 1, 2, \dots, \text{size_col} \end{aligned} \quad (9.2)$$

234 Also for any relaxation formula for off - axis case the p symbol appears. This
235 symbol is connected with r cylindrical coordinate of given node:
236

$$r = ph_r \quad (9.3)$$

237 SO:

$$p == (i_row - 1) \quad (9.4)$$

238 **10 Example of B-type mesh in ANSI C (on axis)**

239 Example of B- type mesh in ANSI C program. The mesh is analogous to A -
 240 type mesh. There are no electric field gradients on mesh borders.

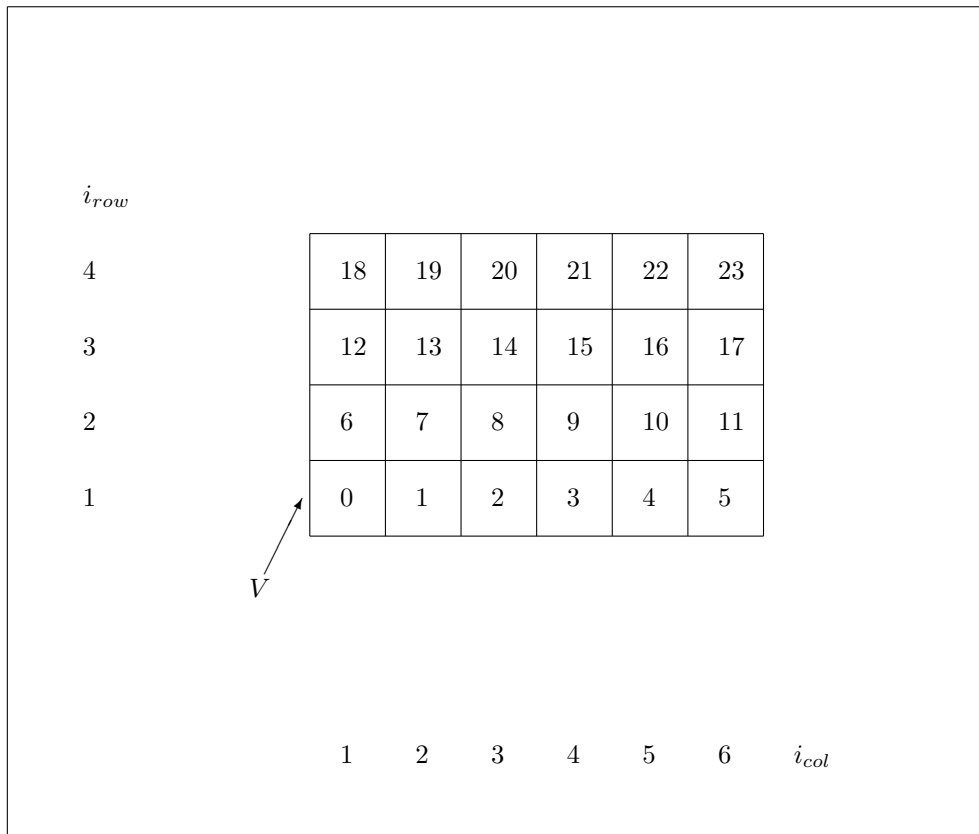


Figure 7: ANSI C - mesh XY type B

- 241 • $V \equiv \text{double* ptr}_V$
- 242 • `unsigned int size_row == 4`
- 243 • `unsigned int size_col == 6`
- 244 • `unsigned int i_row == 1, 2, .., 4`
- 245 • `unsigned int i_col == 1,2, .., 6`
- 246 • `double h_z == 1.0 [mm]`
- 247 • `double h_r == 2.0 [mm]`

248 **11 Example of C-type mesh in ANSI C (on axis)**

249 Example of C- type mesh in ANSI C program. The mesh is analogous to A -
 250 type mesh. Just mesh mesh step $h_x = h_y = h$.

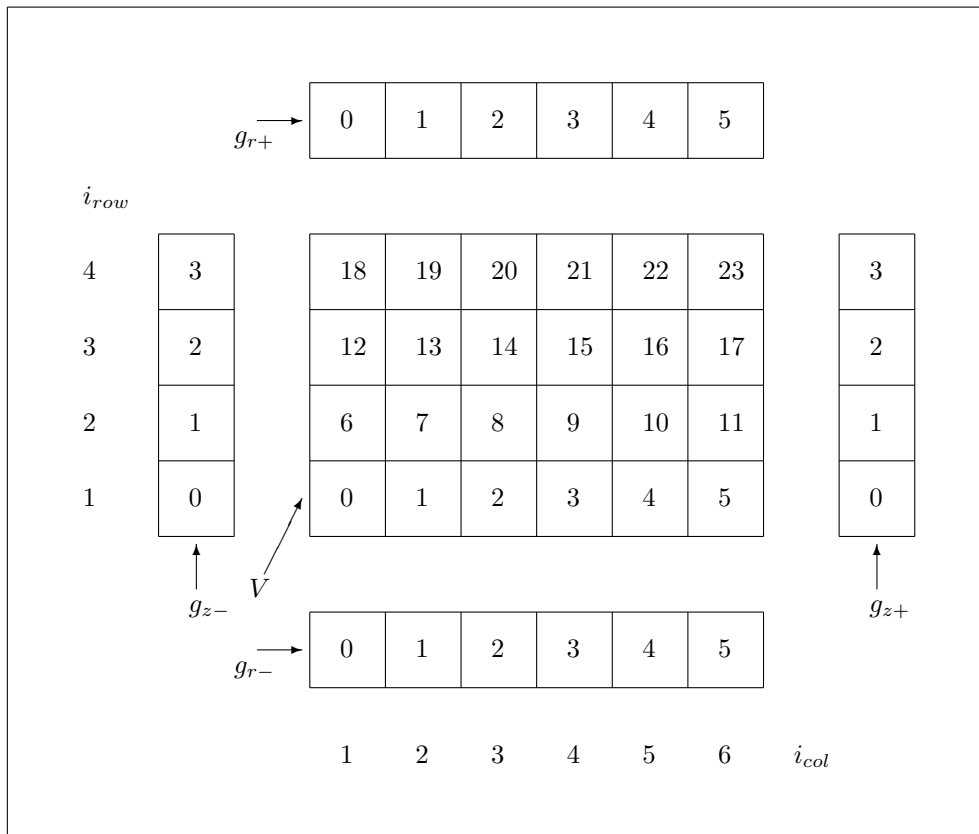


Figure 8: ANSI C - mesh XY type C

251 Note. This is more general example of „off-axis” mesh. If bottom egde of
 252 mesh lies on axis Oz , then gradient g_{r-} does not exist.

- 253 • $g_{z-} \equiv \text{double* ptr_gZ_minus}$
- 254 • $g_{z+} \equiv \text{double* ptr_gZ_plus}$
- 255 • $g_{r-} \equiv \text{double* ptr_gR_minus}$
- 256 • $g_{r+} \equiv \text{double* ptr_gR_plus}$
- 257 • $V \equiv \text{double* ptr_V}$
- 258 • `unsigned int size_row == 4`


```
259 • unsigned int size_col == 6
260 • unsigned int i_row == 1, 2, .., 4
261 • unsigned int i_col == 1,2, .., 6
262 • double h == 1.0 [mm]
```

263 **12 Example of D-type mesh in ANSI C (on axis)**

264 Example of D- type mesh in ANSI C program. The mesh is analogous to B -
265 type mesh. Just $h_x = h_y = h$.

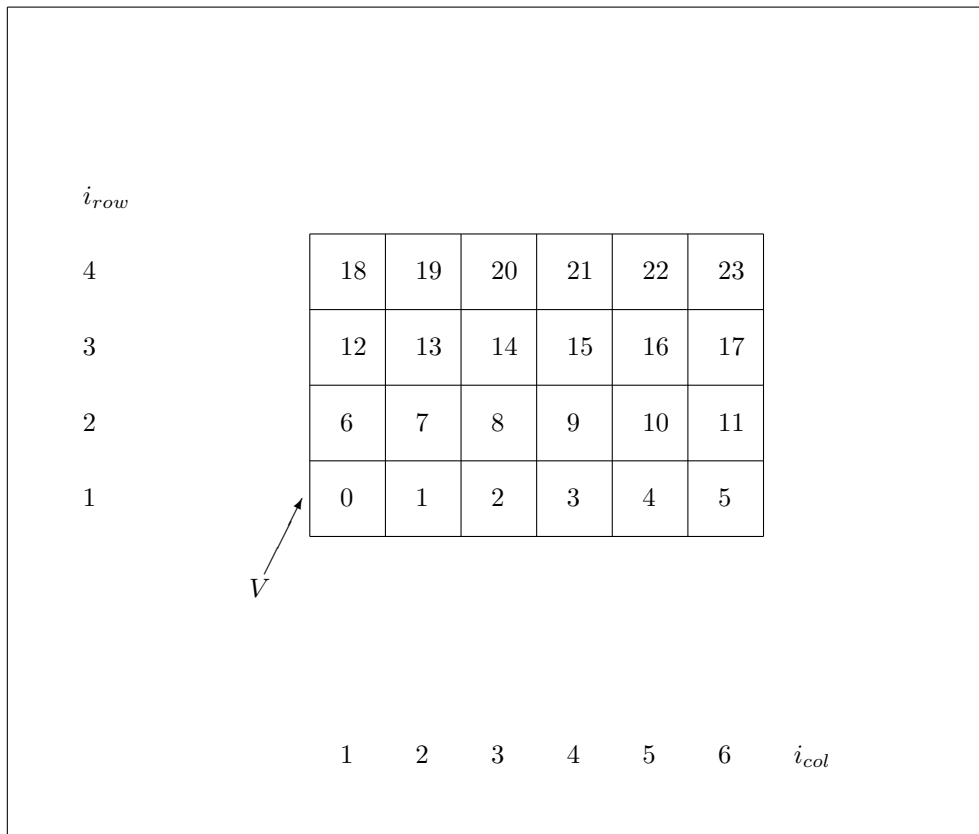


Figure 9: ANSI C - mesh ZR type D

- 266 • $V \equiv \text{double* ptr}_V$
- 267 • `unsigned int size_row == 4`
- 268 • `unsigned int size_col == 6`
- 269 • `unsigned int i_row == 1, 2, .., 4`
- 270 • `unsigned int i_col == 1,2, .., 6`
- 271 • `double h == 1.0 [mm]`

272 **13 Partial derivatives on Oz axis**

273 **13.1 Personal note**

274 This is my personal interpretation. I cannot guarantee correctness of this ap-
275 proach

276 **13.2 Nodes numbering (on axis Oz)**

277 We will try to work with P_2 point (determine approximations of aprtial derivatives
278 for point P_2 , which lies on axis Oz). Nodes numbering on axis Oz differs from
279 numbering convention in Pierre Grivet's book.

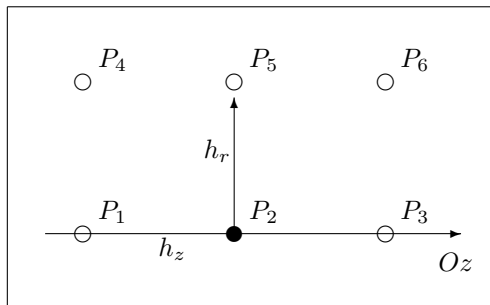


Figure 10: Nodes on axis Oz

280 Point P_2 is situated on Oz axis. It has 2 neighbours on axis Oz : P_1 and P_3 .
281 Node P_5 lies above P_2 node. The mesh step in r direction is h_r . The mesh
282 step in z direction is h_z .

283 **13.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates**

284 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$\begin{aligned}
 V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
 \left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
 \frac{1}{2!} \left(\frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
 \end{aligned} \tag{13.1}$$

285 **13.4 Laplace operator in rotationally symmetrical systems**

286 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3
287 elements [1] (on page 42):

$$\nabla^2 (V_{(z,r)}) = \left(\frac{\partial^2 V}{\partial r^2} \right) + \frac{1}{r} \left(\frac{\partial V}{\partial r} \right) + \left(\frac{\partial^2 V}{\partial z^2} \right) \quad (13.2)$$

288 In this chapter we will try to determine approximation of each term.

289 **13.5 Value of first partial derivative of V with respect to r on axis**
 290 Oz

291 In cylindrically symmetrical field first partial derivative of V (by r) on axis Oz
 292 equals zero (because $V_{(+dr)} = V_{(-dr)}$)

$$\left(\frac{\partial V}{\partial r} \right)_{(z,r=0)} = 0 \quad (13.3)$$

293 **13.6 Value of second partial derivative of V with respect to r on**
 294 **axis Oz**

295 In this subchapter we will try to determine the first term of equation 13.2

296 In our case there is node P_2 on axis Oz . The nearest neighbour of P_2 is
 297 node P_5 , which lies „over Oz axis“. The distance between P_2 and P_5 is h_r .
 298 When we „walk away“ axis Oz in r direction (from point P_2 to point P_5), the
 299 electric potential V_5 can be determined from truncated Taylor expansion 13.1
 300 by expression:

$$V_5 \approx V_2 + \left(\frac{\partial V}{\partial r} \right)_{P_2} \cdot h_r + \frac{1}{2!} \left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} \cdot h_r^2 \quad (13.4)$$

301 We want to determine the second derivative:

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} = ? \quad (13.5)$$

302 We solve equation 13.4 (using relation 13.3).

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_2} \approx \frac{2!(V_5 - V_2)}{h_r^2} = \frac{2(V_5 - V_2)}{h_r^2} \quad (13.6)$$

303 This is final form of approximation of the second derivative of V with respect
 304 to r on axis Oz . It will help us to determine Laplace operator in rotationally
 305 symmetrical systems.

306 **13.7 Value of first partial derivative of V with respect to r divided**
 307 **by r on axis Oz**

308 We will try to determine the second term of relation 13.2 When we are on Oz
 309 axis, the second term has to be determined (because it aims to value $\frac{0}{0}$).

310 When we „ walk away” axis Oz in r direction, the electric potential $V_{(z_0,r)}$
 311 can be determined from truncated Taylor expansion by:

312

$$V_{(z_0,r)} \approx V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) \quad (13.7)$$

313 On Oz axis $r_0 = 0$, so $(r_0 - r) = r$

314

315 Thus we have:

$$V_{(z_0,r)} \approx V_{(z_0,0)} + \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot r \quad (13.8)$$

316 Now let us differentiate (both sides) of such relation:

$$\left| \frac{\partial}{\partial r} \right. \quad (13.9)$$

317 We get:

$$\left(\frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} + \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r + \left(\frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot 1 \quad (13.10)$$

318 On axis Oz we can apply relation 13.3. That's why we can remove these
 319 two terms (first and third) from equation 13.10:

320 So we get (if $r = 0$):

$$\left(\frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r \quad (13.11)$$

321 We can now divide both sides by r .

$$\left| \cdot \frac{1}{r} \right. \quad (13.12)$$

322 We have relation, which has been published in Pierre Grivet's book[1].

$$\left(\frac{1}{r} \frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left(\frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \quad (13.13)$$

323 Approximation of this term on numerical mesh has been already determined
 324 in previous subsection (13.6).

325 **13.8 Value of second partial derivative of V with respect to z on**
326 **axis Oz**

327 The third term of Laplace operator in rotationally symmetrical systems 13.2
328 takes form (on picture 10):

$$\left(\frac{\partial^2 V}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (13.14)$$

329 Now we have determined all the 3 approximations o partial derivatives of V
330 in cylindrically symmetrical systems (on axis O_z).

331 **14 Partial derivatives off O_z axis**

332 **14.1 Personal note**

333 This is my personal interpretation. I cannot guarantee correctness of this ap-
334 proach

335 **14.2 Nodes numbering in Liebmann mesh (off axis O_z)**

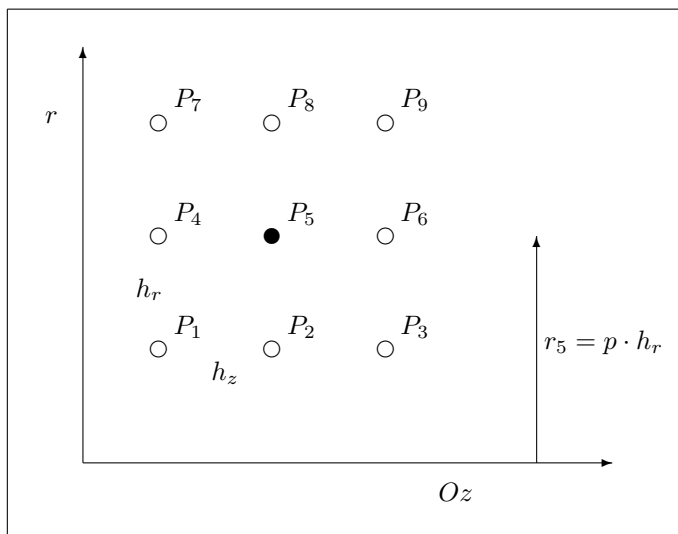


Figure 11: Nodes off axis O_z . Exemplary vector r_5 describes distance from axis O_z to node P_5

336 Mesh step in z direction is h_z . Mesh step in r direction is h_r . Sample mesh
337 points P_5 lies off O_z axis. Distance between mesh point P_5 and O_z axis is r_5 .

338 For ANSI C meshes (in Liebmann source code) the following relations have
339 place:

$$r = p h_r \quad (14.1)$$

$$p = i_{row} - 1 \quad (14.2)$$

340 **14.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates**

341 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates [3]:

$$\begin{aligned}
V_{(z,r)} = V_{(z_0,r_0)} + \left(\frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
\left(\frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
\frac{1}{2!} \left(\frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
\end{aligned} \tag{14.3}$$

342 **14.4 Laplace operator in rotationally symmetrical systems**

343 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3
344 elements [1] (on page 42):

$$\begin{aligned}
\nabla^2 (V_{(z,r)}) = \left(\frac{\partial^2 V}{\partial r^2} \right) + \\
\frac{1}{r} \left(\frac{\partial V}{\partial r} \right) + \\
\left(\frac{\partial^2 V}{\partial z^2} \right)
\end{aligned} \tag{14.4}$$

345 In this chapter we will try to determine approximation of each term.

346 **14.5 Value of second partial derivative of V with respect to r off**
347 **axis Oz**

$$\left(\frac{\partial^2 V}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \tag{14.5}$$

348 **14.6 Value of first partial derivative of V with respect to r divided**
349 **by r off axis Oz**

$$\frac{1}{r_5} \left(\frac{\partial V}{\partial r} \right)_{P_5} \approx \frac{1}{r_5} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2r_5 h_z} \tag{14.6}$$

350 **14.7 Value of second partial derivative of V with respect to z off**
351 **axis Oz**

$$\left(\frac{\partial^2 V}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \tag{14.7}$$

352 **15 Relaxation formula for node P1 (on axis Oz)**

353 **15.1 Node description**

354 Left, bottom corner of mesh ZR (on axis Oz).

355 **15.2 Calculation of relaxation formula**

356 Laplace equation at node P_1

$$\nabla^2 (V_{(z,r)})_{P_1} = 0 \quad (15.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} = 0 \quad (15.2)$$

357 Approximation of partial derivatives of $V_{(z,r)}$ at node P_1

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_r} - \frac{V_1 - V_4}{h_r}}{h_r} = \frac{2(V_4 - V_1)}{h_r^2} \quad (15.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} \approx \frac{2(V_4 - V_1)}{h_r^2} \quad (15.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_z} - \frac{V_1 - V_{1z-}}{h_z}}{h_z} = \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} \quad (15.5)$$

358 Let us substitute approximations to Laplace equation.

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} = 0 \quad (15.6)$$

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} = \frac{g_{1z-}}{h_z} \quad (15.7)$$

359 Let us find V_1

$$V_1 = ? \quad (15.8)$$

360 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (15.9)$$

361 We obtain

$$2V_4 h_z^2 - 2V_1 h_z^2 + 2V_4 h_z^2 - 2V_1 h_z^2 + V_2 h_r^2 - V_1 h_r^2 = g_{1z-} h_z h_r^2 \quad (15.10)$$

362 Let us simplify this equation:

$$V_1 (2h_z^2 + 2h_z^2 + h_r^2) = 2V_4h_z^2 + 2V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \quad (15.11)$$

363 So we have:

$$V_1 (4h_z^2 + h_r^2) = 4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \quad (15.12)$$

364 15.3 Final forms of relaxation formula

365 15.3.1 zrLV_RELAX5_P1_ON_A

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.13)$$

366 15.3.2 zrLV_RELAX5_P1_ON_B

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.14)$$

367 15.3.3 zrLV_RELAX5_P1_ON_C

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4 + V_2 - g_{1z-}h}{5} \end{aligned} \quad (15.15)$$

368 15.3.4 zrLV_RELAX5_P1_ON_D

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4 + V_2}{5} \end{aligned} \quad (15.16)$$

369 **16 Relaxation formula for node P2 (on axis Oz)**

370 **16.1 Node description**

371 Bottom edge of mesh ZR (on axis Oz).

372 **16.2 Calculation of relaxation formula**

373 Laplace equation at node P_2

$$\nabla^2 (V_{(z,r)})_{P_2} = 0 \quad (16.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} = 0 \quad (16.2)$$

374 Approximation of partial derivatives of $V_{(z,r)}$ at node P_2

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_r} - \frac{V_2 - V_5}{h_r}}{h_r} = \frac{2(V_5 - V_2)}{h_r^2} \quad (16.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} \approx \frac{2(V_5 - V_2)}{h_r^2} \quad (16.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (16.5)$$

375 Let us substitute approximations to Laplace equation.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.6)$$

376 There are no g expressions to move, to formula 7 has identical form as
377 formula 6.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.7)$$

378 Let us find V_2

$$V_2 = ? \quad (16.8)$$

379 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (16.9)$$

380 We obtain

$$2V_5 h_z^2 - 2V_2 h_z^2 + 2V_5 h_r^2 - 2V_2 h_r^2 + V_1 h_r^2 + V_3 h_r^2 - 2V_2 h_r^2 = 0 \quad (16.10)$$

381 Let us simplify this equation:

$$V_2 (2h_z^2 + 2h_z^2 + 2h_r^2) = 2V_5 h_z^2 + 2V_5 h_z^2 + V_1 h_r^2 + V_3 h_r^2 \quad (16.11)$$

382 So we have:

$$V_2 (4h_z^2 + 2h_r^2) = 4V_5 h_z^2 + (V_1 + V_3) h_r^2 \quad (16.12)$$

383 16.3 Final forms of relaxation formula

384 16.3.1 zrLV_RELAX5_P2_ON_A

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5 h_z^2 + (V_1 + V_3) h_r^2}{4h_z^2 + 2h_r^2} \quad (16.13)$$

385 16.3.2 zrLV_RELAX5_P2_ON_B

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5 h_z^2 + (V_1 + V_3) h_r^2}{4h_z^2 + 2h_r^2} \quad (16.14)$$

386 16.3.3 zrLV_RELAX5_P2_ON_C

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.15)$$

387 16.3.4 zrLV_RELAX5_P2_ON_D

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.16)$$

388 **17 Relaxation formula for node P3 (on axis Oz)**

389 **17.1 Node description**

390 Right, bottom corner of mesh ZR (on axis Oz).

391 **17.2 Calculation of relaxation formula**

392 Laplace equation at node P_3

$$\nabla^2 (V_{(z,r)})_{P_3} = 0 \quad (17.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} = 0 \quad (17.2)$$

393 Approximation of partial derivatives of $V_{(z,r)}$ at node P_3

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_r} - \frac{V_3 - V_6}{h_r}}{h_r} = \frac{2(V_6 - V_3)}{h_r^2} \quad (17.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} \approx \frac{2(V_6 - V_3)}{h_r^2} \quad (17.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} \approx \frac{\frac{V_{3z+} - V_3}{h_z} - \frac{V_3 - V_2}{h_z}}{h_z} = \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} \quad (17.5)$$

394 Let us substitute approximations to Laplace equation.

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} = 0 \quad (17.6)$$

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} = -\frac{g_{3z+}}{h_z} \quad (17.7)$$

395 Let us find V_3

$$V_3 = ? \quad (17.8)$$

396 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (17.9)$$

397 We obtain

$$2V_6 h_z^2 - 2V_3 h_z^2 + 2V_6 h_z^2 - 2V_3 h_z^2 + V_2 h_r^2 - V_3 h_r^2 = -g_{3z+} h_z h_r^2 \quad (17.10)$$

398 Let us simplify this equation:

$$V_3 (2h_z^2 + 2h_z^2 + h_r^2) = 2V_6h_z^2 + 2V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2 \quad (17.11)$$

399 So we have:

$$V_3 (4h_z^2 + h_r^2) = 4V_6h_z^2 + V_2h_r^2 + g_{1z-}h_zh_r^2 \quad (17.12)$$

400 17.3 Final forms of relaxation formula

401 17.3.1 zrLV_RELAX5_P3_ON_A

$$\begin{aligned} & h_z \neq h_r \\ & g_{3z+} \neq 0 \\ 402 \quad & V_3 = \frac{4V_6h_z^2 + V_2h_r^2 + g_{3z+}h_zh_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.13)$$

403 17.3.2 zrLV_RELAX5_P3_ON_B

$$\begin{aligned} & h_z \neq h_r \\ & g_{3z+} = 0 \\ & V_3 = \frac{4V_6h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.14)$$

404 17.3.3 zrLV_RELAX5_P3_ON_C

$$\begin{aligned} & h_z = h_r = h \\ & g_{3z+} \neq 0 \\ & V_3 = \frac{4V_6 + V_2 + g_{3z+}h}{5} \end{aligned} \quad (17.15)$$

405 17.3.4 zrLV_RELAX5_P3_ON_D

$$\begin{aligned} & h_z = h_r = h \\ & g_{3z+} = 0 \\ & V_3 = \frac{4V_6 + V_2}{5} \end{aligned} \quad (17.16)$$

406 **18 Relaxation formula for node P4**

407 **18.1 Node description**

408 Left edge of mesh ZR.

409 **18.2 Calculation of relaxation formula**

410 Laplace equation at node P_4

$$\nabla^2 (V_{(z,r)})_{P_4} = 0 \quad (18.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} = 0 \quad (18.2)$$

411 Approximation of partial derivatives of $V_{(z,r)}$ at node P_4

412

413 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_r} - \frac{V_4 - V_1}{h_r}}{h_r} = \frac{V_1 + V_7 - 2V_4}{h_r^2} \quad (18.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} \approx \frac{1}{r} \frac{V_7 - V_1}{2h_r} = \frac{V_7 - V_1}{2ph_r^2} \quad (18.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_1}{h_z} - \frac{V_4 - V_{4z-}}{h_z}}{h_z} = \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} \quad (18.5)$$

414 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} = 0 \quad (18.6)$$

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} = \frac{g_{4z-}}{h_z} \quad (18.7)$$

415 Let us find V_4

$$V_4 = ? \quad (18.8)$$

416 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (18.9)$$

417 We obtain

$$2pV_1h_z^2 + 2pV_7h_z^2 - 4pV_4h_z^2 + V_7h_z^2 - V_1h_z^2 + 2pV_5h_r^2 - 2pV_4h_r^2 = 2pg_{4z-}h_zh_r^2 \quad (18.10)$$

418 Let us simplify this equation:

$$V_4(4ph_z^2 + 2ph_r^2) = V_1(2ph_z^2 - h_z^2) + V_7(2ph_z^2 + h_z^2) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.11)$$

419 So we have:

$$V_42p(2h_z^2 + h_r^2) = V_1h_z^2(2p-1) + V_7h_z^2(2p+1) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.12)$$

420 18.3 Final forms of relaxation formula

421 18.3.1 zrLV_RELAX5_P4_A

$$h_z \neq h_r$$

$$g_{4z-} \neq 0$$

$$422 \quad V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 - \frac{g_{4z-}h_zh_r^2}{2h_z^2 + h_r^2} \quad (18.13)$$

423 18.3.2 zrLV_RELAX5_P4_B

$$h_z \neq h_r$$

$$g_{4z-} = 0$$

$$424 \quad V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (18.14)$$

425 18.3.3 zrLV_RELAX5_P4_C

$$h_z = h_r = h$$

$$g_{4z-} \neq 0$$

$$426 \quad V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 - \frac{g_{4z-}h}{3} \quad (18.15)$$

427 18.3.4 zrLV_RELAX5_P4_D

$$h_z = h_r = h$$

$$g_{4z-} = 0$$

$$V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 \quad (18.16)$$

428 **19 Relaxation formula for node P5**

429 **19.1 Node description**

430 Inner node of mesh ZR.

431 **19.2 Calculation of relaxation formula**

432 Laplace equation at node P_5

$$\nabla^2 (V_{(z,r)})_{P_5} = 0 \quad (19.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} = 0 \quad (19.2)$$

433 Approximation of partial derivatives of $V_{(z,r)}$ at node P_5

434

435 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \quad (19.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} \approx \frac{1}{r} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2ph_r^2} \quad (19.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \quad (19.5)$$

436 Let us substitute approximations to Laplace equation.

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.6)$$

437 We don't need to simplify this equation in step 7:

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.7)$$

438 Let us find V_5

$$V_5 = ? \quad (19.8)$$

439 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (19.9)$$

440 We obtain

$$2pV_2h_z^2 + 2pV_8h_z^2 - 4pV_5h_z^2 + V_8h_z^2 - V_2h_z^2 + 2pV_4h_r^2 + 2pV_6h_r^2 - 4pV_5h_r^2 = 0 \quad (19.10)$$

441 Let us simplify this equation:

$$V_5(4ph_z^2 + 4ph_r^2) = V_2(2ph_z^2 - h_z^2) + V_8(2ph_z^2 + h_z^2) + 2ph_r^2V_4 + 2ph_r^2V_6 \quad (19.11)$$

442 So we have:

$$V_54p(h_z^2 + h_r^2) = V_2h_z^2(2p - 1) + V_8h_z^2(2p + 1) + V_42ph_r^2 + V_62ph_r^2 \quad (19.12)$$

443 19.3 Final forms of relaxation formula

444 19.3.1 zrLV_RELAX5_P5_A

$$445 \quad h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.13)$$

446 19.3.2 zrLV_RELAX5_P5_B

447 This formula is identical to formula A:

$$448 \quad h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.14)$$

449 19.3.3 zrLV_RELAX5_P5_C

$$450 \quad h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.15)$$

451 19.3.4 zrLV_RELAX5_P5_D

452 This formula is identical to formula C:

$$453 \quad h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.16)$$

454 **20 Relaxation formula for node P6**

455 **20.1 Node description**

456 Right edge of mesh ZR.

457 **20.2 Calculation of relaxation formula**

458 Laplace equation at node P_6

$$\nabla^2 (V_{(z,r)})_{P_6} = 0 \quad (20.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} = 0 \quad (20.2)$$

459 Approximation of partial derivatives of $V_{(z,r)}$ at node P_6

460

461 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} \approx \frac{\frac{V_9 - V_6}{h_r} - \frac{V_6 - V_3}{h_r}}{h_r} = \frac{V_3 + V_9 - 2V_6}{h_r^2} \quad (20.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} \approx \frac{1}{r} \frac{V_9 - V_3}{2h_r} = \frac{V_9 - V_3}{2ph_r^2} \quad (20.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} \approx \frac{\frac{V_{6z+} - V_6}{h_z} - \frac{V_6 - V_5}{h_z}}{h_z} = \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} \quad (20.5)$$

462 Let us substitute approximations to Laplace equation.

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} = 0 \quad (20.6)$$

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} = -\frac{g_{6z+}}{h_z} \quad (20.7)$$

463 Let us find V_6

$$V_6 = ? \quad (20.8)$$

464 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (20.9)$$

465 We obtain

$$2pV_3h_z^2 + 2pV_9h_z^2 - 4pV_6h_z^2 + V_9h_z^2 - V_3h_z^2 + 2pV_5h_r^2 - 2pV_6h_r^2 = -2pg_{6z+}h_zh_r^2 \quad (20.10)$$

466 Let us simplify this equation:

$$V_6(4ph_z^2 + 2ph_r^2) = V_3(2ph_z^2 - h_z^2) + V_9(2ph_z^2 + h_z^2) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.11)$$

467 So we have:

$$V_62p(2h_z^2 + h_r^2) = V_3h_z^2(2p-1) + V_9h_z^2(2p+1) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.12)$$

468 20.3 Final forms of relaxation formula

469 20.3.1 zrLV_RELAX5_P6_A

$$h_z \neq h_r$$

$$g_{6z+} \neq 0$$

$$470 \quad V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 + \frac{g_{6z+}h_zh_r^2}{2h_z^2 + h_r^2} \quad (20.13)$$

471 20.3.2 zrLV_RELAX5_P6_B

$$h_z \neq h_r$$

$$g_{6z+-} = 0$$

$$472 \quad V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (20.14)$$

473 20.3.3 zrLV_RELAX5_P6_C

$$h_z = h_r = h$$

$$g_{6z+} \neq 0$$

$$474 \quad V_6 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 + \frac{g_{6z+}h}{3} \quad (20.15)$$

475 20.3.4 zrLV_RELAX5_P6_D

$$h_z = h_r = h$$

$$g_{6z+} = 0$$

$$476 \quad V_4 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 \quad (20.16)$$

477 **21 Relaxation formula for node P7**

478 **21.1 Node description**

479 Left, upper corner of mesh ZR.

480 **21.2 Calculation of relaxation formula**

481 Laplace equation at node P_7

$$\nabla^2 (V_{(z,r)})_{P_7} = 0 \quad (21.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} = 0 \quad (21.2)$$

482 Approximation of partial derivatives of $V_{(z,r)}$ at node P_7

483

484 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} \approx \frac{\frac{V_{7r+} - V_7}{h_r} - \frac{V_7 - V_4}{h_r}}{h_r} = \frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} \quad (21.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} \approx \frac{1}{r} \frac{V_{7r+} - V_4}{2h_r} = \frac{V_7 + g_{7r+}h_r - V_4}{2ph_r^2} = \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} \quad (21.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_z} - \frac{V_7 - V_{7z-}}{h_z}}{h_z} = \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} \quad (21.5)$$

485 Let us substitute approximations to Laplace equation.

$$\frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} + \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} + \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} = 0 \quad (21.6)$$

$$\frac{V_4 - V_7}{h_r^2} + \frac{V_7 - V_4}{2ph_r^2} + \frac{V_8 - V_7}{h_z^2} = -\frac{g_{7r+}}{h_r} - \frac{g_{7r+}}{2ph_r} + \frac{g_{7z-}}{h_z} \quad (21.7)$$

486 Let us find V_7

$$V_7 = ? \quad (21.8)$$

487 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (21.9)$$

488 We obtain

$$2pV_4 h_z^2 - 2pV_7 h_z^2 + V_7 h_z^2 - V_4 h_z^2 + 2pV_8 h_r^2 - 2pV_7 h_r^2 = \quad (21.10)$$

$$-2pg_{7r+} h_z^2 h_r - g_{7r+} h_z^2 h_r + 2pg_{7z-} h_z h_r^2$$

489 Let us simplify this equation:

$$V_7 (2ph_z^2 - h_z^2 + 2ph_r^2) = V_4 (2ph_z^2 - h_z^2) + V_8 (2ph_r^2) + \quad (21.11)$$

$$2pg_{7r+} h_z^2 h_r + g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2$$

490 So we have:

$$V_7 ((2p-1)h_z^2 + 2ph_r^2) = V_4 h_z^2 (2p-1) + V_8 2ph_r^2 + \quad (21.12)$$

$$2pg_{7r+} h_z^2 h_r + g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2$$

491 21.3 Final forms of relaxation formula

492 21.3.1 zrLV_RELAX5_P7_A

$$h_z \neq h_r$$

$$g_{7z-} \neq 0$$

$$g_{7r+} \neq 0$$

493

$$V_7 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 + \quad (21.13)$$

$$\frac{(2p+1)g_{7r+} h_z^2 h_r - 2pg_{7z-} h_z h_r^2}{(2p-1)h_z^2 + 2ph_r^2}$$

494 21.3.2 zrLV_RELAX5_P7_B

$$h_z \neq h_r$$

$$g_{7z-} = 0$$

$$g_{7r+} = 0$$

$$V_7 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 \quad (21.14)$$

495 **21.3.3 zrLV_RELAX5_P7_C**

$$h_z = h_r = h$$

$$g_{7z-} \neq 0$$

$$g_{7r+} \neq 0$$

496

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{7r+} - g_{7z-})}{4p-1} \quad (21.15)$$

497 **21.3.4 zrLV_RELAX5_P7_D**

$$h_z = h_r = h$$

$$g_{7z-} = 0$$

$$g_{7r+} = 0$$

498

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 \quad (21.16)$$

499 **22 Relaxation formula for node P8**

500 **22.1 Node description**

501 Upper edge of mesh ZR.

502 **22.2 Calculation of relaxation formula**

503 Laplace equation at node P_8

$$\nabla^2 (V_{(z,r)})_{P_8} = 0 \quad (22.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} = 0 \quad (22.2)$$

504 Approximation of partial derivatives of $V_{(z,r)}$ at node P_8

505

506 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} \approx \frac{\frac{V_{8r+} - V_8}{h_r} - \frac{V_8 - V_5}{h_r}}{h_r} = \frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} \quad (22.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} \approx \frac{1}{r} \frac{V_{8r+} - V_5}{2h_r} = \frac{V_8 + g_{8r+}h_r - V_5}{2ph_r^2} = \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} \quad (22.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_z} - \frac{V_8 - V_7}{h_z}}{h_z} = \frac{V_7 + V_9 - 2V_8}{h_z^2} \quad (22.5)$$

507 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} + \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = 0 \quad (22.6)$$

$$\frac{V_5 - V_8}{h_r^2} + \frac{V_8 - V_5}{2ph_r^2} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = -\frac{g_{8r+}}{h_r} - \frac{g_{8r+}}{2ph_r} \quad (22.7)$$

508 Let us find V_8

$$V_8 = ? \quad (22.8)$$

509 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (22.9)$$

510 We obtain

$$2pV_5 h_z^2 - 2pV_8 h_z^2 + V_8 h_z^2 - V_5 h_z^2 + 2pV_7 h_r^2 + 2pV_9 h_r^2 - 4pV_8 h_r^2 = -2pg_{8r+} h_z^2 h_r - g_{8r+} h_z^2 h_r \quad (22.10)$$

511 Let us simplify this equation:

$$V_8 (2ph_z^2 - h_z^2 + 4ph_r^2) = V_5 (2ph_z^2 - h_z^2) + (V_7 + V_9) 2ph_r^2 + 2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r \quad (22.11)$$

512 So we have:

$$V_8 ((2p-1)h_z^2 + 4ph_r^2) = V_5 h_z^2 (2p-1) + (V_7 + V_9) 2ph_r^2 + 2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r \quad (22.12)$$

513 22.3 Final forms of relaxation formula

514 22.3.1 zrLV_RELAX5_P8_A

$$h_z \neq h_r$$

$$g_{8r+} \neq 0$$

515

$$V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) + \frac{(2p+1)h_z^2 h_r g_{8r+}}{(2p-1)h_z^2 + 4ph_r^2} \quad (22.13)$$

516 22.3.2 zrLV_RELAX5_P8_B

$$h_z \neq h_r$$

$$g_{8r+} = 0$$

517

$$V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) \quad (22.14)$$

518 22.3.3 zrLV_RELAX5_P8_C

$$h_z = h_r = h$$

$$g_{8r+} \neq 0$$

519

$$V_8 = \frac{2p-1}{6p-1} V_5 + \frac{2p}{6p-1} (V_7 + V_9) + \frac{(2p+1)h g_{8r+}}{6p-1} \quad (22.15)$$

520 **22.3.4 zrLV_RELAX5_P8_D**

$$h_z = h_r = h$$

$$g_{8r+} = 0$$

521

$$V_8 = \frac{2p-1}{6p-1}V_5 + \frac{2p}{6p-1}(V_7 + V_9) \quad (22.16)$$

522 **23 Relaxation formula for node P9**

523 **23.1 Node description**

524 Right, upper corner of mesh ZR.

525 **23.2 Calculation of relaxation formula**

526 Laplace equation at node P_9

$$\nabla^2 (V_{(z,r)})_{P_9} = 0 \quad (23.1)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} + \left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} + \left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} = 0 \quad (23.2)$$

527 Approximation of partial derivatives of $V_{(z,r)}$ at node P_9

528

529 (note: $r = ph_r$ (such as in Pierre Grivet's book))

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} \approx \frac{V_{9r+} - V_9}{h_r} - \frac{V_9 - V_6}{h_r} = \frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} \quad (23.3)$$

$$\left(\frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} \approx \frac{1}{r} \frac{V_{9r+} - V_6}{2h_r} = \frac{V_9 + g_{9r+}h_r - V_6}{2ph_r^2} = \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} \quad (23.4)$$

$$\left(\frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} \approx \frac{V_{9z+} - V_9}{h_z} - \frac{V_9 - V_8}{h_z} = \frac{g_{9z+}}{h_z} + \frac{V_8 - V_9}{h_z^2} \quad (23.5)$$

530 Let us substitute approximations to Laplace equation.

$$\frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} + \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} + \frac{V_8 - V_9}{h_z^2} + \frac{g_{9z+}}{h_z} = 0 \quad (23.6)$$

$$\frac{V_6 - V_9}{h_r^2} + \frac{V_9 - V_6}{2ph_r^2} + \frac{V_8 - V_9}{h_z^2} = -\frac{g_{9r+}}{h_r} - \frac{g_{9r+}}{2ph_r} - \frac{g_{9z+}}{h_z} \quad (23.7)$$

531 Let us find V_9

$$V_9 = ? \quad (23.8)$$

532 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (23.9)$$

533 We obtain

$$2pV_6 h_z^2 - 2pV_9 h_z^2 + V_9 h_z^2 - V_6 h_z^2 + 2pV_8 h_r^2 - 2pV_9 h_r^2 = \quad (23.10)$$

$$-2pg_{9r+} h_z^2 h_r - g_{9r+} h_z^2 h_r - 2pg_{9z+} h_z h_r^2$$

534 Let us simplify this equation:

$$V_9 (2ph_z^2 - h_z^2 + 2ph_r^2) = V_6 (2ph_z^2 - h_z^2) + V_8 (2ph_r^2) + \quad (23.11)$$

$$2pg_{9r+} h_z^2 h_r + g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2$$

535 So we have:

$$V_9 ((2p-1)h_z^2 + 2ph_r^2) = V_6 h_z^2 (2p-1) + V_8 2ph_r^2 + \quad (23.12)$$

$$(2p+1)g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2$$

536 23.3 Final forms of relaxation formula

537 23.3.1 zrLV_RELAX5_P9_A

$$h_z \neq h_r$$

$$g_{9z-} \neq 0$$

$$g_{9r+} \neq 0$$

538

$$V_9 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 + \quad (23.13)$$

$$\frac{(2p+1)g_{9r+} h_z^2 h_r + 2pg_{9z+} h_z h_r^2}{(2p-1)h_z^2 + 2ph_r^2}$$

539 23.3.2 zrLV_RELAX5_P9_B

$$h_z \neq h_r$$

$$g_{9z-} = 0$$

$$g_{9r+} = 0$$

$$V_9 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 \quad (23.14)$$

540 **23.3.3 zrLV_RELAX5_P9_C**

$$h_z = h_r = h$$

$$g_{9z-} \neq 0$$

$$g_{9r+} \neq 0$$

541

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{9r+} + g_{9z+})}{4p-1} \quad (23.15)$$

542 **23.3.4 zrLV_RELAX5_P9_D**

$$h_z = h_r = h$$

$$g_{9z-} = 0$$

$$g_{9r+} = 0$$

543

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 \quad (23.16)$$

544 **References**

545 [1] P. Grivet, *Electron Optics, Second (revised) English edition*. Pergamon
546 Press Ltd., 1972.

547 [2] J. R. Nagel, "Solving the generalized poisson equation using the fi-
548 nite - difference method (fdm)." [https://my.ece.utah.edu/~ece6340/](https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/)
549 [LECTURES/Feb1/](https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/), 2012. [Online; accessed 2-February-2023].

550 [3] kryomaxim, "taylor expansion in cylindrical coordinates."
551 [https://math.stackexchange.com/questions/1133311/](https://math.stackexchange.com/questions/1133311/taylor-expansion-in-cylindrical-coordinates)
552 [taylor-expansion-in-cylindrical-coordinates](https://math.stackexchange.com/questions/1133311/taylor-expansion-in-cylindrical-coordinates), 2015. [Online;
553 accessed 9-May-2023].

554 [4] A. Septier(ed.), *Focusing of Charged Paticles. Volume I*. New York and
555 London, Academic Press, 1967.

556 [5] A. Septier(ed.), *Applied Charged Paticle Optics, part A*. New York and Lon-
557 don, Academic Press, 1980.

558 [6] D. W. O. Heddle, *Electrostatic Lens Systems. Second Edition*. Institute of
559 Physics Publishing, Bristol and Philadelphia, 2000.

560 [7] B. Paszkowski, *Optyka Elektronowa, wydanie II, poprawione i uzupełnione*.
561 Państwowe Wydawnictwa Naukowo - Techniczne, Warszawa, 1965.

562 [8] B. Paszkowski, *Electron Optics [by] B.Paszkowski. Translated from the Pol-
563 ish by George Lepa. English translation edited by R. C. G. Leckey*. London,
564 Iliffe; New York, American Elsevier Publishing Company Inc., 1968.