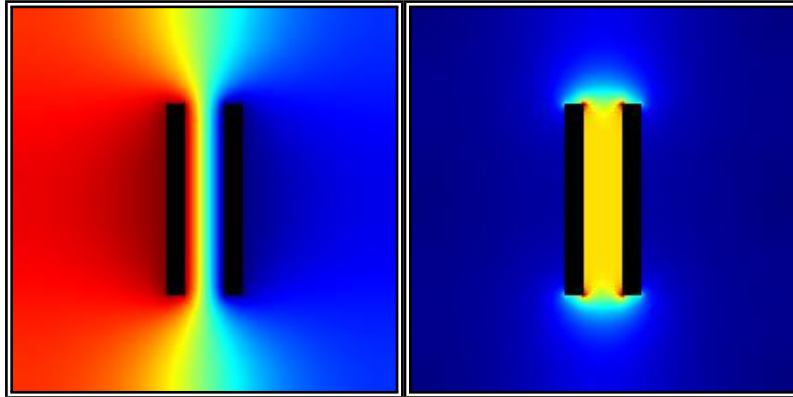


1

Liebmann technical documentation



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3

Laplace equation 2D (XY)
(Cartesian coordinates)
relaxation scheme explained
(5 - point star)

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version 11

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2024.12.13

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University of Maria Curie - Skłodowska in Lublin, Poland

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99 **1 Liebmann technical documentation series**

- 100 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-
101 sacyjną Liebmann. (Polish version / wersja polska)
- 102 2. Determination of electrostatic field distribution by using Liebmann relax-
103 ation method. (English version / wersja angielska)
- 104 3. Graphics. Mapping voltages to colours (colormaps).
- 105 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme
106 explained. (5 - point star)
- 107 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme
108 explained. (5 - point star)
- 109 6. Liebmann source code. (ANSI C programming language)

110 **2 Versions of this document**

- 111 1. version 1 - 2023.11.03
- 112 2. version 2 - 2024.01.26
- 113 3. version 3 - 2024.02.02
- 114 4. version 4 - 2024.02.05
- 115 5. version 5 - 2024.05.18
- 116 6. version 6 - 2024.05.23
- 117 7. version 7 - 2024.05.24
- 118 8. version 8 - 2024.07.17
- 119 9. version 9 - 2024.07.18
- 120 10. version 10 - 2024.09.03
- 121 11. version 11 - 2024.12.13

122 **3 Solving Laplace equation using relaxation method**

123 I tried to solve Laplace equation using mainly information from Pierre Grivet's
124 book (Electron Optics) - [1].
125 There are few editions of this book (1965, 1972). Second edition (1972) con-
126 tains explanation of relaxation method (page 38).

127 More generalized approaches has been drafted by James R. Nagel - [2].
128 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).

129

130 There are also publications edited by Albert Septier: Focusing of Charged
131 Particles [3] and Applied Charged Particle Optics (part A). [4].

132 I have also found some ideas in publication of D W O Heddle: Electrostatic
133 Lens Systems [5] (especially using PC computers to solve electrostatic prob-
134 lems).

135 I have also found (brief) description of by - hand solving of Laplace equa-
136 tion by Bohdan Paszkowski - [6] (Polish edition). English translation of this book
137 also exists - [7].

138

139 I would like to thank many people, who helped me with this challenge. Espe-
140 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),
141 who enabled me to use SIMION and MATLAB software while writing master's
142 thesis about electron optical systems at University of Maria Curie - Skłodowska
143 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-
144 sion about numerical methods. What is more, my colleague Bartosz in 2012
145 had explained me general problems with software efficiency. So he had also
146 contributed significantly to the idea of Liebmann software (especially using C
147 language).

148 4 Explanation of symbols in calculations

149 • P_i - i -th mesh node

150 • V_i - value of electrostatic potential at node P_i . Unit - [V]

151 • h - mesh step (for example h_x - mesh step in x direction). Unit - [mm]

152 • $g_{i+/-}$ - gradient in direction i (for example $g_{1x-} = \frac{V_1 - V_{1x-}}{h_x}$. Unit - [$\frac{V}{mm}$])

153 • i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, \dots, \text{size_row}$

154 • i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, \dots, \text{size_col}$

155 Symbols in final relaxation formulae

156 xyLV_RELAX5_P1_A

157 • xy - coordinates (2D, planar)

158 • LV - Laplace equation in vacuum (no dielectrics)

159 • RELAX5 - 5- point relaxation method

160 • P1 - relaxation scheme for point P1 (in general P1 .. P9)

161 • A - mesh type A (in general A .. D)

162 **5 Mesh XY - type A**

163 $h_x \neq h_y$

164 gradient V outside a mesh exists

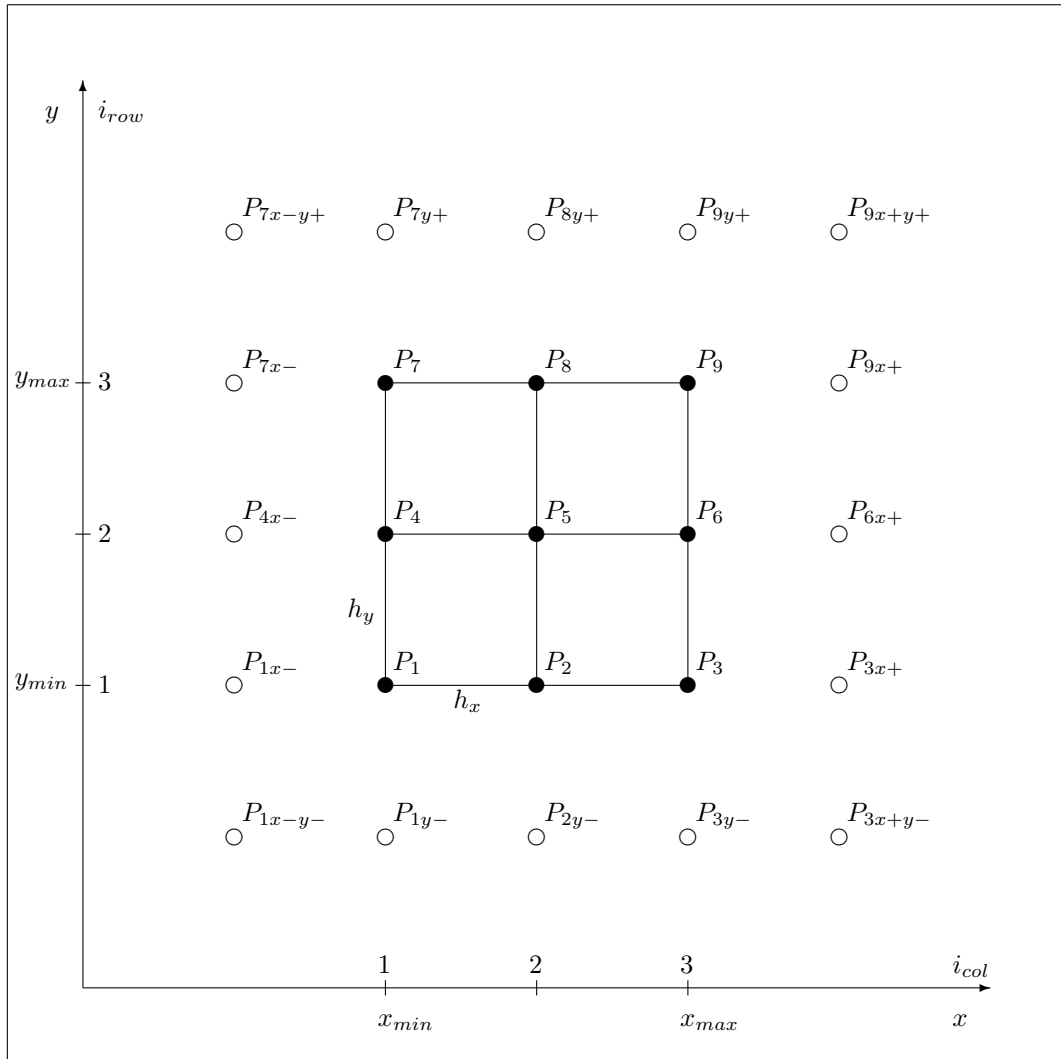


Figure 1: Mesh XY type A

165 **6 Mesh XY - type B**

166 $h_x \neq h_y$

167 gradient V outside a mesh does not exist

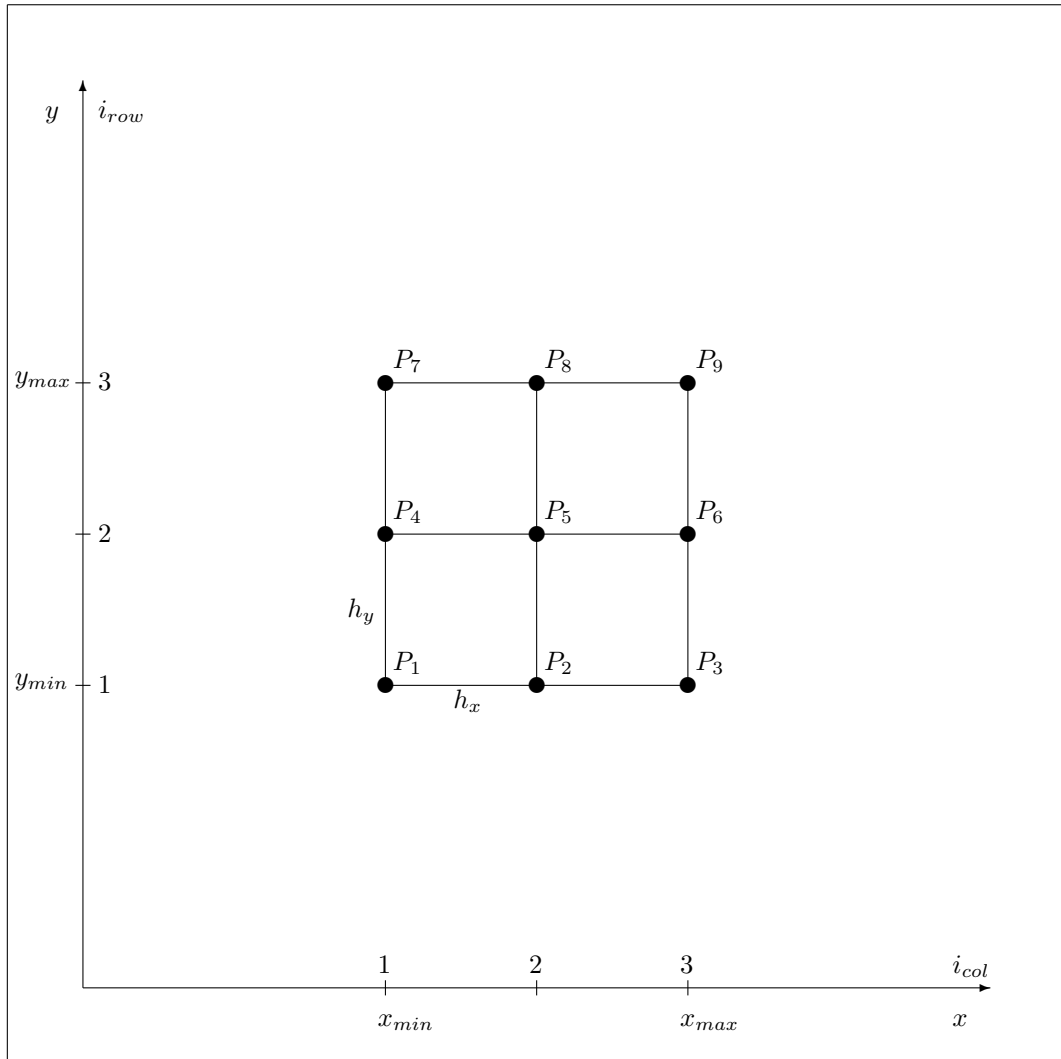


Figure 2: Mesh XY type B

168 **7 Mesh XY - type C**

169 $h_x = h_y = h$

170 gradient V outside a mesh exists

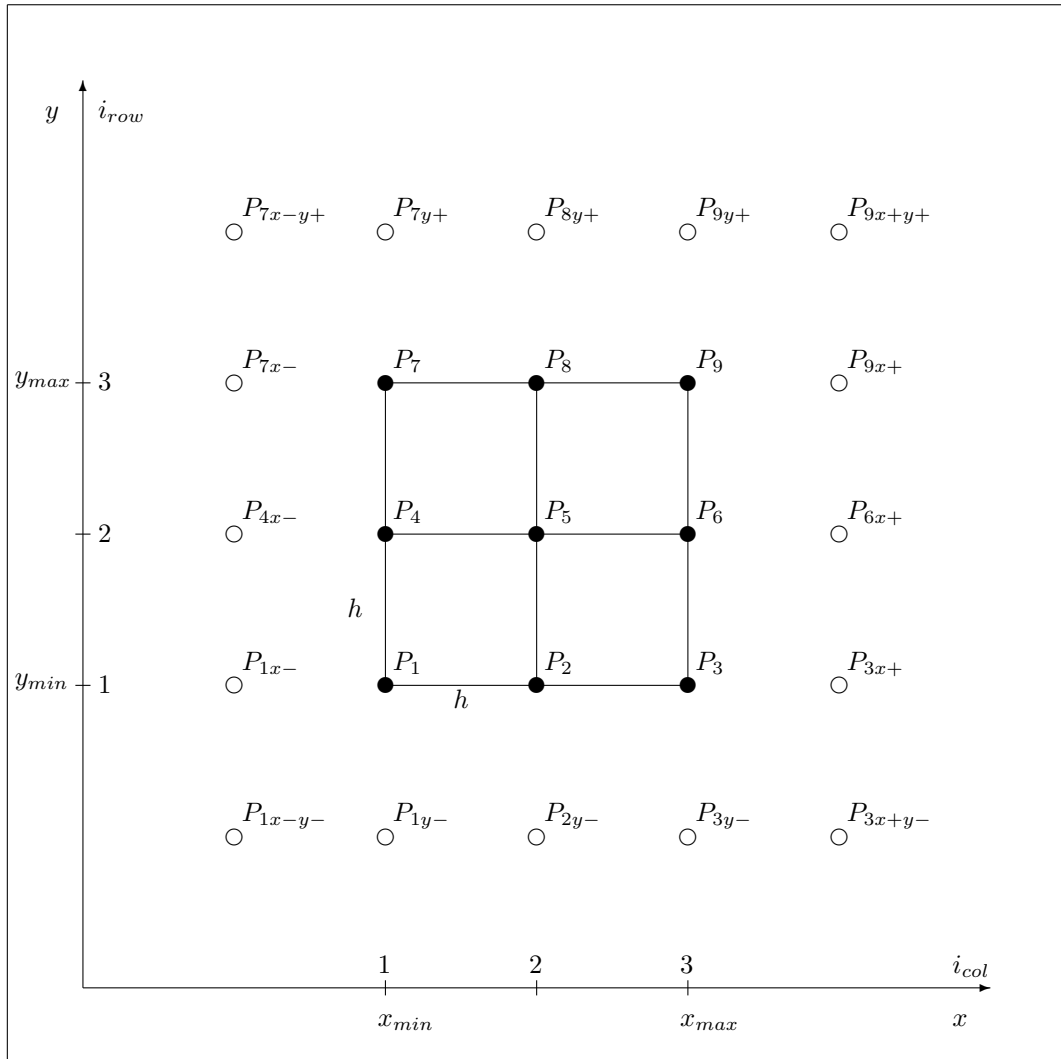


Figure 3: Mesh XY type C

171 **8 Mesh XY - type D**

172 $h_x = h_y = h$

173 gradient V outside a mesh does not exist

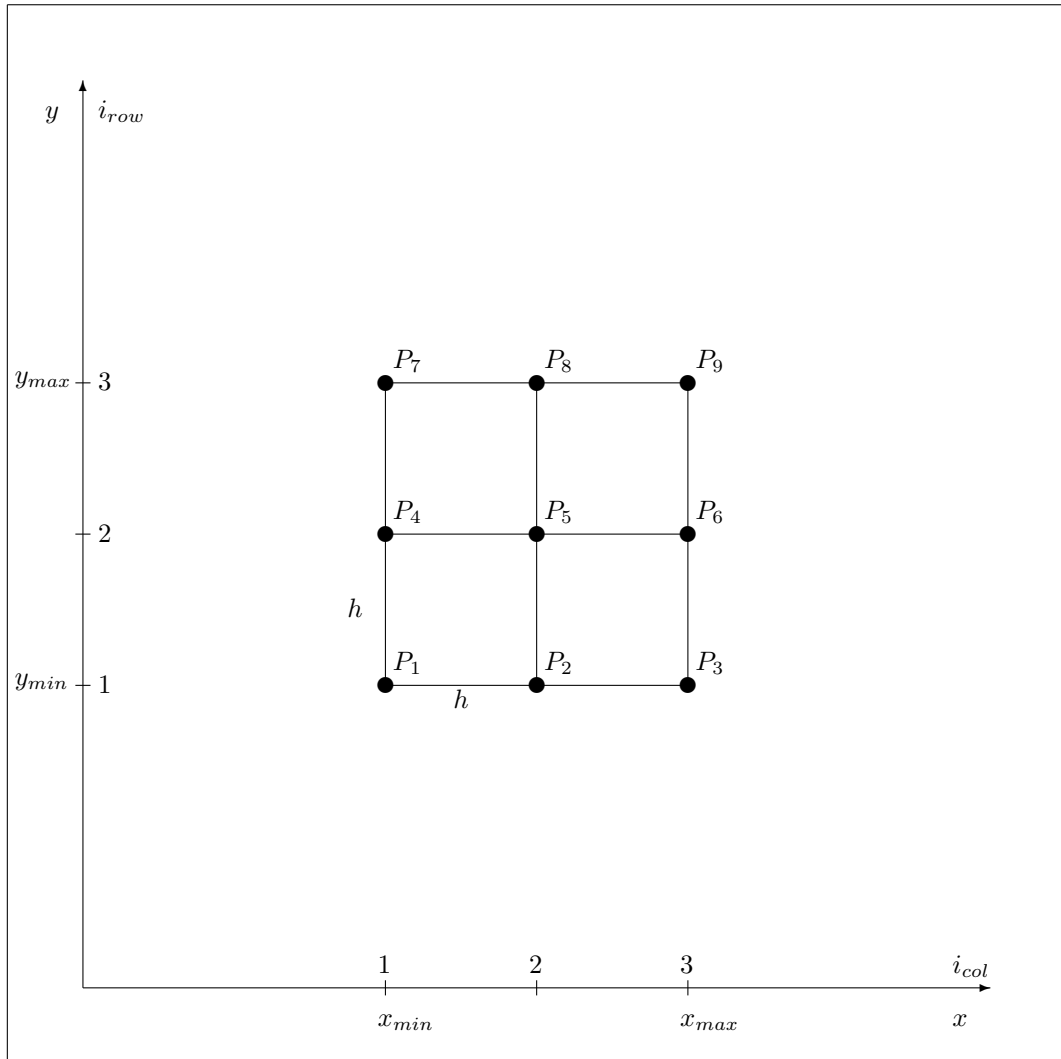


Figure 4: Mesh XY type D

174 **9 Example of A-type mesh in ANSI C**

175 Example of A- type mesh in ANSI C program. The mesh is represented by 2
 176 dimensional array of double precision numbers. Rows and columns in mesh
 177 are numbered from 1 (this was my choice) instead of default 0 (as usual in C
 178 language). This choice has pros and cons. It is easier to calculate mesh size
 179 (size_row * size_col). Access to each node can be also more intuitive, but logic
 180 in each library function must contain this shift between node ordering styles.

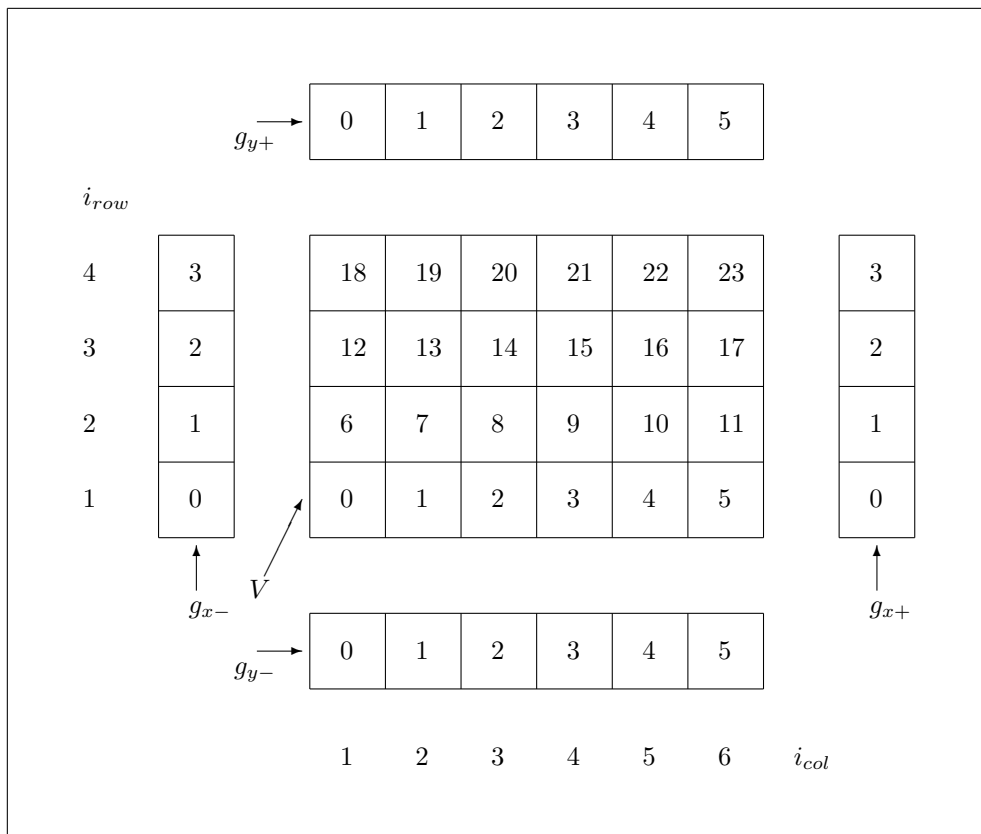
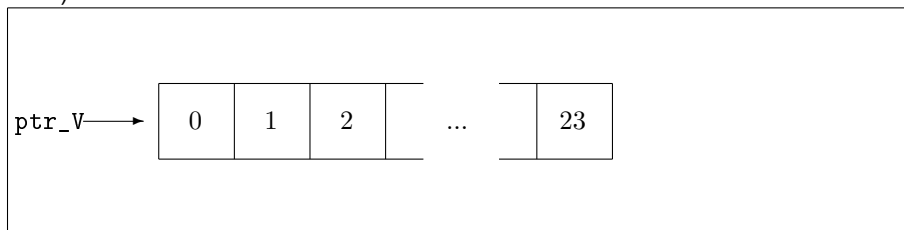


Figure 5: ANSI C - mesh XY type A

- 181 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 182 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 183 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 184 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 185 • $V \equiv \text{double* ptr_V}$
- 186 • `unsigned int size_row == 4`

- 187 • unsigned int size_col == 6
- 188 • unsigned int i_row == 1, 2, .., 4
- 189 • unsigned int i_col == 1,2, .., 6
- 190 • double h_x == 1.0 [mm]
- 191 • double h_y == 2.0 [mm]

192 The following picture describes analogous version of ptr_V mesh, which
 193 can be dynamically allocated on heap by pointer method. The mesh is rep-
 194 resented by single block of memory. The numbers or rows and columns are
 195 also known, so each node can be also accessed by appropriate index (memory
 196 address).



197
 198 Each mesh point has its unique index (let's say icp - (index of central
 199 point)), which can be determined, if we know indices of row and column (i_row,
 200 i_col).

$$icp == (i_row - 1) * size_col + i_col - 1 \quad (9.1)$$

201 For example for each point of a mesh indices of row and column have val-
 202 ues:

$$\begin{aligned} i_row &== 1, 2, \dots, size_row \\ i_col &== 1, 2, \dots, size_col \end{aligned} \quad (9.2)$$

203 **10 Example of B-type mesh in ANSI C**

204 Example of B- type mesh in ANSI C program. The mesh is analogous to A -
205 type mesh. There are no electric field gradients on mesh borders.

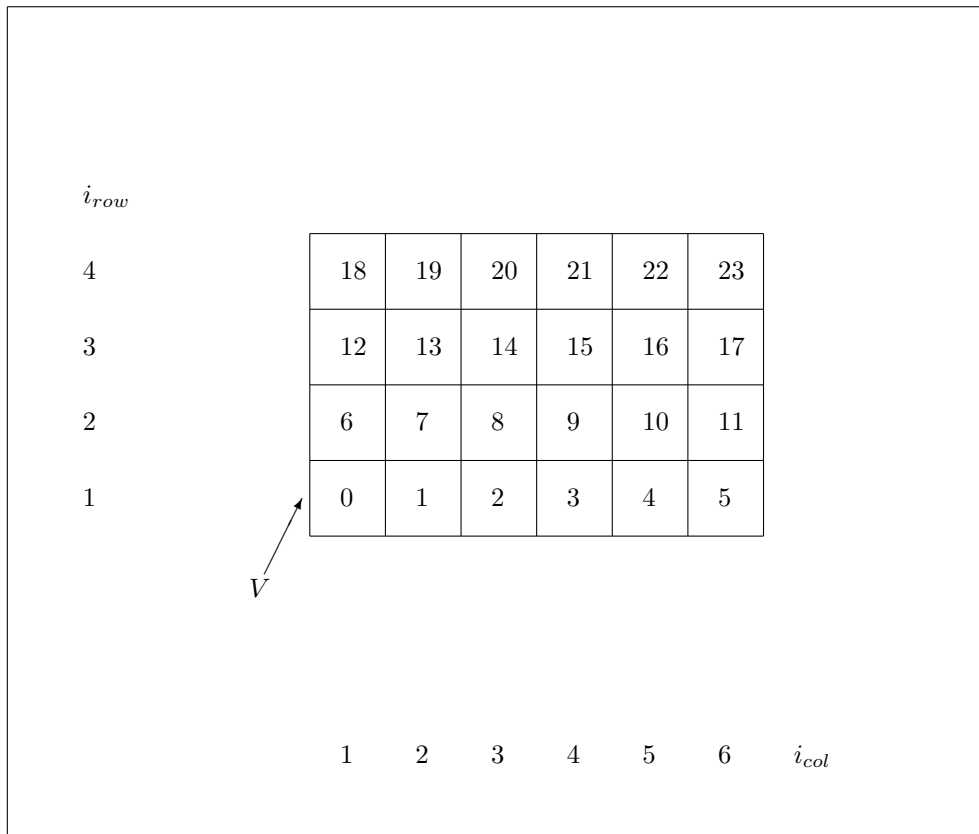


Figure 6: ANSI C - mesh XY type B

- 206 • $V \equiv \text{double* ptr}_V$
- 207 • `unsigned int size_row == 4`
- 208 • `unsigned int size_col == 6`
- 209 • `unsigned int i_row == 1, 2, ..., 4`
- 210 • `unsigned int i_col == 1,2, ..., 6`
- 211 • `double h_x == 1.0 [mm]`
- 212 • `double h_y == 2.0 [mm]`

213 **11 Example of C-type mesh in ANSI C**

214 Example of C- type mesh in ANSI C program. The mesh is analogous to A -
 215 type mesh. Just mesh mesh step $h_x = h_y = h$.

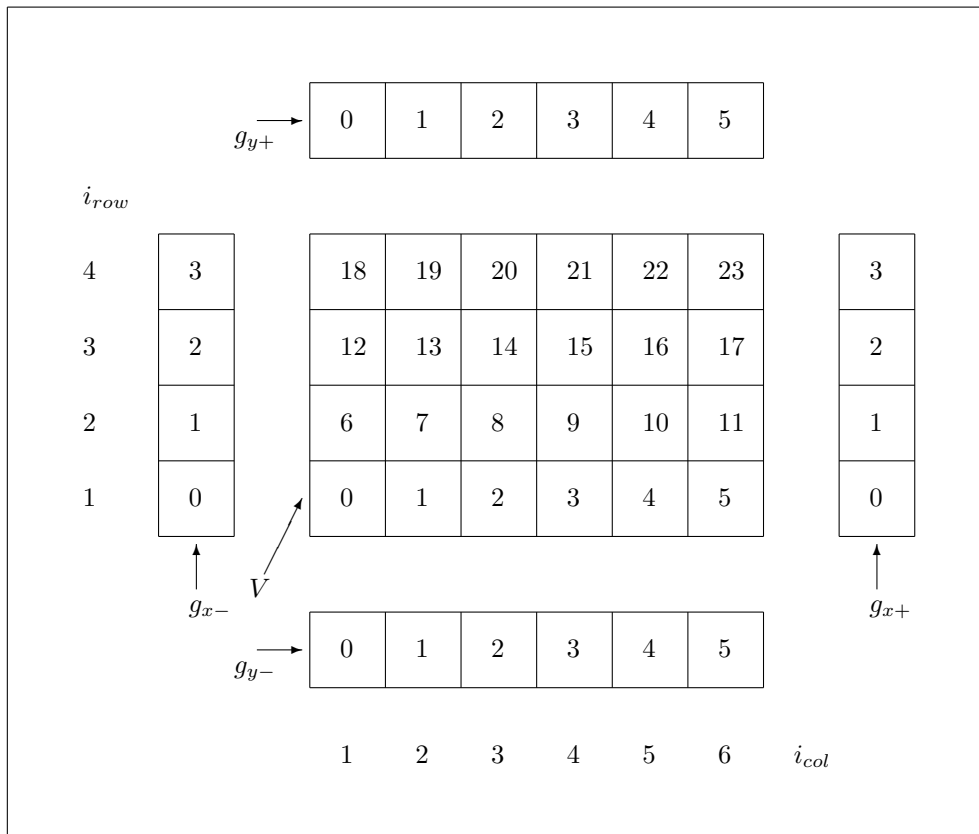


Figure 7: ANSI C - mesh XY type C

- 216 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 217 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 218 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 219 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 220 • $V \equiv \text{double* ptr_V}$
- 221 • `unsigned int size_row == 4`
- 222 • `unsigned int size_col == 6`
- 223 • `unsigned int i_row == 1, 2, .., 4`

```
224 • unsigned int i_col == 1,2, .., 6
225 • double h == 1.0 [mm]
```

226 **12 Example of D-type mesh in ANSI C**

227 Example of D- type mesh in ANSI C program. The mesh is analogous to B -
228 type mesh. Just $h_x = h_y = h$.

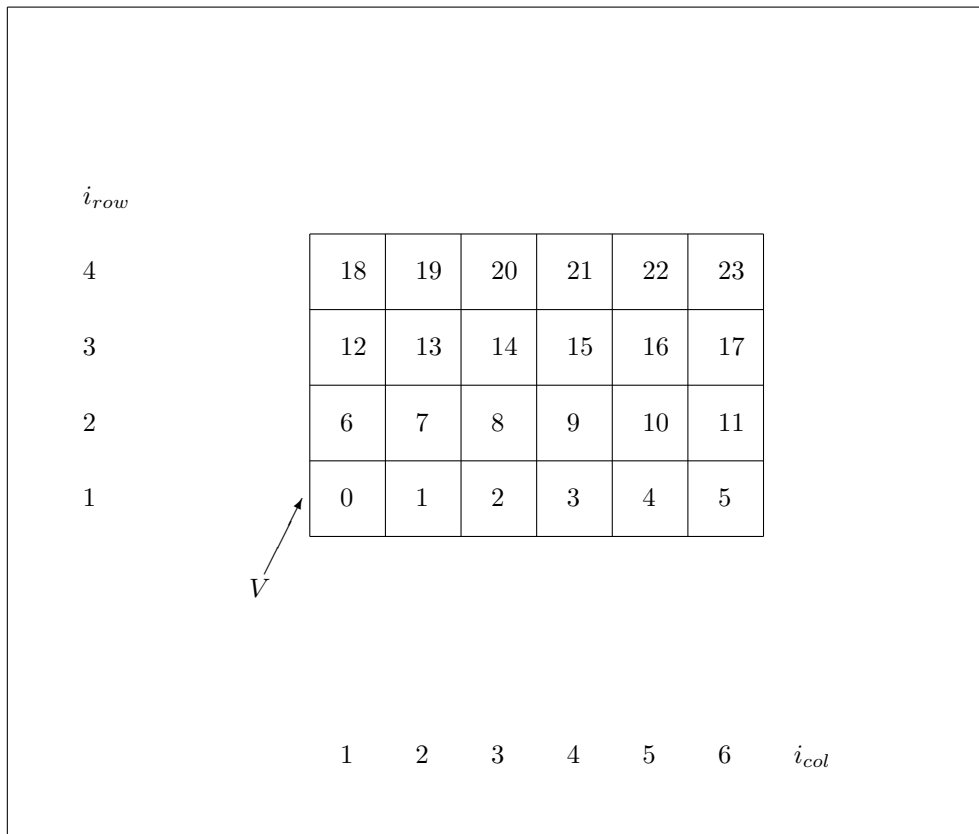


Figure 8: ANSI C - mesh XY type D

- 229 • $V \equiv \text{double* ptr}_V$
- 230 • `unsigned int size_row == 4`
- 231 • `unsigned int size_col == 6`
- 232 • `unsigned int i_row == 1, 2, ..., 4`
- 233 • `unsigned int i_col == 1,2, ..., 6`
- 234 • `double h == 1.0 [mm]`

235 **13 Relaxation formula for node P1**

236 **13.1 Node description**

237 Left, bottom corner of mesh XY.

238 **13.2 Calculation of relaxation formula**

239 Laplace equation at node P_1

$$\nabla^2 (V_{(x,y)})_{P_1} = 0 \quad (13.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} = 0 \quad (13.2)$$

240 Approximation of partial derivatives of $V_{(x,y)}$ at node P_1

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_x} - \frac{V_1 - V_{1y-}}{h_x}}{h_x} = \frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} \quad (13.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_y} - \frac{V_1 - V_{1y-}}{h_y}}{h_y} = \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} \quad (13.4)$$

241 Let us substitute approximations to Laplace equation.

$$\frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} + \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} = 0 \quad (13.5)$$

242 Let us find V_1

$$V_1 = ? \quad (13.6)$$

$$\frac{V_2 - V_1}{h_x^2} + \frac{V_4 - V_1}{h_y^2} = \frac{g_{1x-}}{h_x} + \frac{g_{1y-}}{h_y} \quad (13.7)$$

243 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (13.8)$$

244 We obtain

$$V_2 h_y^2 - V_1 h_y^2 + V_4 h_x^2 - V_1 h_x^2 = g_{1x-} h_x h_y^2 + g_{1y-} h_x^2 h_y \quad (13.9)$$

$$V_1 (h_x^2 + h_y^2) = V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y \quad (13.10)$$

245 **13.3 Final forms of relaxation formula**

246 **13.3.1 xyLV_RELAX5_P1_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} \neq 0 \\ 247 \quad V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (13.11)$$

248 **13.3.2 xyLV_RELAX5_P1_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (13.12)$$

249 **13.3.3 xyLV_RELAX5_P1_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} \neq 0 \\ V_1 &= \frac{V_2 + V_4 - g_{1x-} h - g_{1y-} h}{2} \end{aligned} \quad (13.13)$$

250 **13.3.4 xyLV_RELAX5_P1_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 + V_4}{2} \end{aligned} \quad (13.14)$$

251 **14 Relaxation formula for node P2**

252 **14.1 Node description**

253 Bottom edge of mesh XY.

254 **14.2 Calculation of relaxation formula**

255 Laplace equation at node P_2

$$\nabla^2 (V_{(x,y)})_{P_2} = 0 \quad (14.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} = 0 \quad (14.2)$$

256 Approximation of partial derivatives of $V_{(x,y)}$ at node P_2

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_x} - \frac{V_2 - V_1}{h_x}}{h_x} = \frac{V_1 + V_3 - 2V_2}{h_x^2} \quad (14.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_y} - \frac{V_2 - V_{2y-}}{h_y}}{h_y} = \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} \quad (14.4)$$

257 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} = 0 \quad (14.5)$$

258 Let us find V_2

$$V_2 = ? \quad (14.6)$$

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} = \frac{g_{2y-}}{h_y} \quad (14.7)$$

259 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (14.8)$$

260 We obtain

$$V_1 h_y^2 + V_3 h_y^2 - 2V_2 h_y^2 + V_5 h_x^2 = g_{2y-} h_x^2 h_y \quad (14.9)$$

$$V_2 (h_x^2 + h_y^2) = (V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y \quad (14.10)$$

261 **14.3 Final forms of relaxation formula**

262 **14.3.1 xyLV_RELAX5_P2_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (14.11)$$

263 **14.3.2 xyLV_RELAX5_P2_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &= 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (14.12)$$

264 **14.3.3 xyLV_RELAX5_P2_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{V_1 + V_3 + V_5 - g_{2y-} h}{3} \end{aligned} \quad (14.13)$$

265 **14.3.4 xyLV_RELAX5_P2_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &= 0 \\ V_2 &= \frac{V_1 + V_3 + V_5}{3} \end{aligned} \quad (14.14)$$

266 **15 Relaxation formula for node P3**

267 **15.1 Node description**

268 Right, bottom corner of mesh XY.

269 **15.2 Calculation of relaxation formula**

270 Laplace equation at node P_3

$$\nabla^2 (V_{(x,y)})_{P_3} = 0 \quad (15.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} = 0 \quad (15.2)$$

271 Approximation of partial derivatives of $V_{(x,y)}$ at node P_3

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} \approx \frac{V_{3x+} - V_3 - \frac{V_3 - V_2}{h_x}}{h_x} = \frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} \quad (15.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_y} - \frac{V_3 - V_{3y-}}{h_y}}{h_y} = \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} \quad (15.4)$$

272 Let us substitute approximations to Laplace equation.

$$\frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} = 0 \quad (15.5)$$

273 Let us find V_3

$$V_3 = ? \quad (15.6)$$

$$\frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} = \frac{g_{3y-}}{h_y} - \frac{g_{3x+}}{h_x} \quad (15.7)$$

274 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (15.8)$$

275 We obtain

$$V_2 h_y^2 - V_3 h_y^2 + V_6 h_x^2 - V_3 h_x^2 = g_{3y-} h_x^2 h_y - g_{3x+} h_x h_y^2 \quad (15.9)$$

$$V_3 (h_x^2 + h_y^2) = V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y \quad (15.10)$$

276 **15.3 Final forms of relaxation formula**

277 **15.3.1 xyLV_RELAX5_P3_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (15.11)$$

278 **15.3.2 xyLV_RELAX5_P3_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (15.12)$$

279 **15.3.3 xyLV_RELAX5_P3_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 + V_6 + g_{3x+} h - g_{3y-} h}{2} \end{aligned} \quad (15.13)$$

280 **15.3.4 xyLV_RELAX5_P3_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 + V_6}{2} \end{aligned} \quad (15.14)$$

281 **16 Relaxation formula for node P4**

282 **16.1 Node description**

283 Left edge of mesh XY.

284 **16.2 Calculation of relaxation formula**

285 Laplace equation at node P_4

$$\nabla^2 (V_{(x,y)})_{P_4} = 0 \quad (16.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} = 0 \quad (16.2)$$

286 Approximation of partial derivatives of $V_{(x,y)}$ at node P_4

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_4}{h_x} - \frac{V_4 - V_{4x-}}{h_x}}{h_x} = \frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} \quad (16.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_y} - \frac{V_4 - V_1}{h_y}}{h_y} = \frac{V_1 + V_7 - 2V_4}{h_y^2} \quad (16.4)$$

287 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = 0 \quad (16.5)$$

288 Let us find V_4

$$V_4 = ? \quad (16.6)$$

$$\frac{V_5 - V_4}{h_x^2} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = \frac{g_{4x-}}{h_x} \quad (16.7)$$

289 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (16.8)$$

290 We obtain

$$V_5 h_y^2 - V_4 h_y^2 + V_1 h_x^2 + V_7 h_x^2 - 2V_4 h_x^2 = g_{4x-} h_x h_y^2 \quad (16.9)$$

$$V_4 (2h_x^2 + h_y^2) = (V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2 \quad (16.10)$$

291 **16.3 Final forms of relaxation formula**

292 **16.3.1 xyLV_RELAX5_P4_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.11)$$

293 **16.3.2 xyLV_RELAX5_P4_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &= 0 \\ V_2 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.12)$$

294 **16.3.3 xyLV_RELAX5_P4_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{V_1 + V_5 + V_7 - g_{4x-} h}{3} \end{aligned} \quad (16.13)$$

295 **16.3.4 xyLV_RELAX5_P4_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &= 0 \\ V_4 &= \frac{V_1 + V_5 + V_7}{3} \end{aligned} \quad (16.14)$$

296 **17 Relaxation formula for node P5**

297 **17.1 Node description**

298 Node inside a mesh XY.

299 **17.2 Calculation of relaxation formula**

300 Laplace equation at node P_5

$$\nabla^2 (V_{(x,y)})_{P_5} = 0 \quad (17.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} = 0 \quad (17.2)$$

301 Approximation of partial derivatives of $V_{(x,y)}$ at node P_5

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_x} - \frac{V_5 - V_4}{h_x}}{h_x} = \frac{V_4 + V_6 - 2V_5}{h_x^2} \quad (17.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_y} - \frac{V_5 - V_2}{h_y}}{h_y} = \frac{V_2 + V_8 - 2V_5}{h_y^2} \quad (17.4)$$

302 Let us substitute approximations to Laplace equation.

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.5)$$

303 Let us find V_5

$$V_5 = ? \quad (17.6)$$

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.7)$$

304 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (17.8)$$

305 We obtain

$$V_4 h_y^2 + V_6 h_y^2 - 2V_5 h_y^2 + V_2 h_x^2 + V_8 h_x^2 - 2V_5 h_x^2 = 0 \quad (17.9)$$

$$2V_5 (h_x^2 + h_y^2) = (V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2 \quad (17.10)$$

306 **17.3 Final forms of relaxation formula**

307 **17.3.1 xyLV_RELAX5_P5_A**

$$h_x \neq h_y$$

308 No gradients g inside mesh are considered.

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.11)$$

309 **17.3.2 xyLV_RELAX5_P5_B**

$$h_x \neq h_y$$

310 Relaxation formula is the same as xyLV_RELAX5_P5_A

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.12)$$

311 **17.3.3 xyLV_RELAX5_P5_C**

$$h_x = h_y = h$$

312 No gradients g inside mesh are considered.

313 The formula simplifies, so no g and h terms are necessary.

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.13)$$

314 **17.3.4 xyLV_RELAX5_P5_D**

$$h_x = h_y = h$$

315 The formula also simplifies.

316

317 Relaxation formula is the same as xyLV_RELAX5_P5_C

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.14)$$

318 **18 Relaxation formula for node P6**

319 **18.1 Node description**

320 Right edge of mesh XY.

321 **18.2 Calculation of relaxation formula**

322 Laplace equation at node P_6

$$\nabla^2 (V_{(x,y)})_{P_6} = 0 \quad (18.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} = 0 \quad (18.2)$$

323 Approximation of partial derivatives of $V_{(x,y)}$ at node P_6

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} \approx \frac{\frac{V_{6x+} - V_6}{h_x} - \frac{V_6 - V_5}{h_x}}{h_x} = \frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} \quad (18.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} \approx \frac{\frac{V_9 - V_6}{h_y} - \frac{V_6 - V_3}{h_y}}{h_y} = \frac{V_3 + V_9 - 2V_6}{h_y^2} \quad (18.4)$$

324 Let us substitute approximations to Laplace equation.

$$\frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = 0 \quad (18.5)$$

325 Let us find V_6

$$V_6 = ? \quad (18.6)$$

$$\frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = -\frac{g_{6x+}}{h_x} \quad (18.7)$$

326 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (18.8)$$

327 We obtain

$$V_5 h_y^2 - V_6 h_y^2 + V_3 h_x^2 + V_9 h_x^2 - 2V_6 h_x^2 = -g_{6x+} h_x h_y^2 \quad (18.9)$$

$$V_6 (2h_x^2 + h_y^2) = (V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2 \quad (18.10)$$

328 **18.3 Final forms of relaxation formula**

329 **18.3.1 xyLV_RELAX5_P6_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.11)$$

330 **18.3.2 xyLV_RELAX5_P6_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &= 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.12)$$

331 **18.3.3 xyLV_RELAX5_P6_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{V_3 + V_5 + V_9 + g_{6x+} h}{3} \end{aligned} \quad (18.13)$$

332 **18.3.4 xyLV_RELAX5_P6_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &= 0 \\ V_6 &= \frac{V_3 + V_5 + V_9}{3} \end{aligned} \quad (18.14)$$

333 **19 Relaxation formula for node P7**

334 **19.1 Node description**

335 Left, upper corner of mesh XY.

336 **19.2 Calculation of relaxation formula**

337 Laplace equation at node P_7

$$\nabla^2 (V_{(x,y)})_{P_7} = 0 \quad (19.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} = 0 \quad (19.2)$$

338 Approximation of partial derivatives of $V_{(x,y)}$ at node P_7

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_x} - \frac{V_7 - V_{7x-}}{h_x}}{h_x} = \frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} \quad (19.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} \approx \frac{\frac{V_{7y+} - V_7}{h_y} - \frac{V_7 - V_4}{h_y}}{h_y} = \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} \quad (19.4)$$

339 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} + \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} = 0 \quad (19.5)$$

340 Let us find V_7

$$V_7 = ? \quad (19.6)$$

$$\frac{V_8 - V_7}{h_x^2} + \frac{V_4 - V_7}{h_y^2} = \frac{g_{7x-}}{h_x} - \frac{g_{7y+}}{h_y} \quad (19.7)$$

341 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (19.8)$$

342 We obtain

$$V_8 h_y^2 - V_7 h_y^2 + V_4 h_x^2 - V_7 h_x^2 = g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.9)$$

$$V_7 (h_x^2 + h_y^2) = V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.10)$$

343 **19.3 Final forms of relaxation formula**

344 **19.3.1 xyLV_RELAX5_P7_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 + g_{7y+} h_x^2 h_y}{(h_x^2 + h_y^2)} \end{aligned} \quad (19.11)$$

345 **19.3.2 xyLV_RELAX5_P7_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (19.12)$$

346 **19.3.3 xyLV_RELAX5_P7_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 + V_8 - g_{7x-} h + g_{7y+} h}{2} \end{aligned} \quad (19.13)$$

347 **19.3.4 xyLV_RELAX5_P7_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 + V_8}{2} \end{aligned} \quad (19.14)$$

348 **20 Relaxation formula for node P8**

349 **20.1 Node description**

350 Upper edge of mesh XY.

351 **20.2 Calculation of relaxation formula**

352 Laplace equation at node P_8

$$\nabla^2 (V_{(x,y)})_{P_8} = 0 \quad (20.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} = 0 \quad (20.2)$$

353 Approximation of partial derivatives of $V_{(x,y)}$ at node P_8

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_x} - \frac{V_8 - V_7}{h_x}}{h_x} = \frac{V_7 + V_9 - 2V_8}{h_x^2} \quad (20.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} \approx \frac{\frac{V_{8y+} - V_8}{h_y} - \frac{V_8 - V_5}{h_y}}{h_y} = \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} \quad (20.4)$$

354 Let us substitute approximations to Laplace equation.

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} = 0 \quad (20.5)$$

355 Let us find V_8

$$V_8 = ? \quad (20.6)$$

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} = -\frac{g_{8y+}}{h_y} \quad (20.7)$$

356 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (20.8)$$

357 We obtain

$$V_7 h_y^2 + V_9 h_y^2 - 2V_8 h_y^2 + V_5 h_x^2 - V_8 h_x^2 = -g_{8y+} h_x^2 h_y \quad (20.9)$$

$$V_8 (h_x^2 + 2h_y^2) = (V_7 + V_9) h_y^2 + V_5 h_x^2 + g_{8y+} h_x^2 h_y \quad (20.10)$$

358 **20.3 Final forms of relaxation formula**

359 **20.3.1 xyLV_RELAX5_P8_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2 + g_{8y+} h_x^2 h_y}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.11)$$

360 **20.3.2 xyLV_RELAX5_P8_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.12)$$

361 **20.3.3 xyLV_RELAX5_P8_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 + V_7 + V_9 + g_{8y+} h}{3} \end{aligned} \quad (20.13)$$

362 **20.3.4 xyLV_RELAX5_P8_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 + V_7 + V_9}{3} \end{aligned} \quad (20.14)$$

363 **21 Relaxation formula for node P9**

364 **21.1 Node description**

365 Right, upper corner of mesh XY.

366 **21.2 Calculation of relaxation formula**

367 Laplace equation at node P_9

$$\nabla^2 (V_{(x,y)})_{P_9} = 0 \quad (21.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} = 0 \quad (21.2)$$

368 Approximation of partial derivatives of $V_{(x,y)}$ at node P_9

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} \approx \frac{\frac{V_{9x+} - V_9}{h_x} - \frac{V_9 - V_8}{h_x}}{h_x} = \frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} \quad (21.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} \approx \frac{\frac{V_{9y+} - V_9}{h_y} - \frac{V_9 - V_6}{h_y}}{h_y} = \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} \quad (21.4)$$

369 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} + \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} = 0 \quad (21.5)$$

370 Let us find V_9

$$V_9 = ? \quad (21.6)$$

$$\frac{V_8 - V_9}{h_x^2} + \frac{V_8 - V_9}{h_x^2} = -\frac{g_{9x+}}{h_x} - \frac{g_{9y+}}{h_y} \quad (21.7)$$

371 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (21.8)$$

372 We obtain

$$V_8 h_y^2 - V_9 h_y^2 + V_6 h_x^2 - V_9 h_x^2 = -g_{9x+} h_x h_y^2 - g_{9y+} h_x^2 h_y \quad (21.9)$$

$$V_9 (h_x^2 + h_y^2) = V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y \quad (21.10)$$

373 **21.3 Final forms of relaxation formula**

374 **21.3.1 xyLV_RELAX5_P9_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (21.11)$$

375 **21.3.2 xyLV_RELAX5_P9_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (21.12)$$

376 **21.3.3 xyLV_RELAX5_P9_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 + V_8 + g_{9x+} h + g_{9y+} h}{2} \end{aligned} \quad (21.13)$$

377 **21.3.4 xyLV_RELAX5_P9_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 + V_8}{2} \end{aligned} \quad (21.14)$$

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