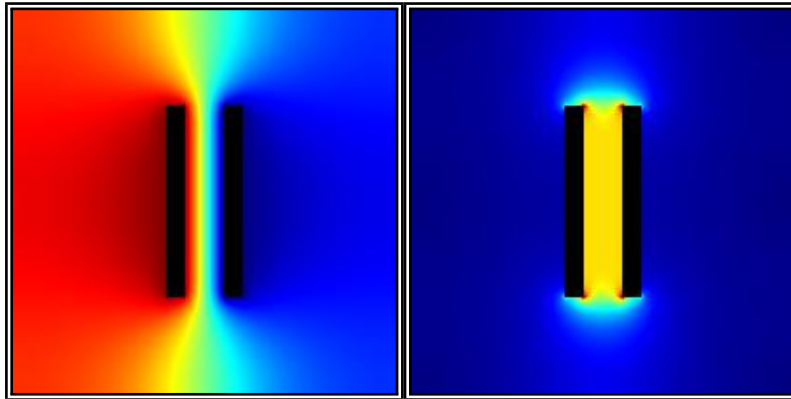


1

Liebmann technical documentation



2

3

Laplace equation 2D (XY)
(Cartesian coordinates)
relaxation scheme explained
(5 - point star)

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project homepage: http://marcinkulbaka.prv.pl/Liebmann/index_en.html

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version 12

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2025.03.13

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University of Maria Curie - Skłodowska in Lublin, Poland

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99 **1 Liebmann technical documentation series**

- 100 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-
101 sacyjną Liebmann. (Polish version / wersja polska)
- 102 2. Determination of electrostatic field distribution by using Liebmann relax-
103 ation method. (English version / wersja angielska)
- 104 3. Graphics. Mapping voltages to colours (colormaps).
- 105 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme
106 explained. (5 - point star)
- 107 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme
108 explained. (5 - point star)
- 109 6. Liebmann source code. (ANSI C programming language)

110 **2 Versions of this document**

- 111 1. version 1 - 2023.11.03
- 112 2. version 2 - 2024.01.26
- 113 3. version 3 - 2024.02.02
- 114 4. version 4 - 2024.02.05
- 115 5. version 5 - 2024.05.18
- 116 6. version 6 - 2024.05.23
- 117 7. version 7 - 2024.05.24
- 118 8. version 8 - 2024.07.17
- 119 9. version 9 - 2024.07.18
- 120 10. version 10 - 2024.09.03
- 121 11. version 11 - 2024.12.13
- 122 12. version 12 - 2025.03.13

123 **3 Solving Laplace equation using relaxation method**

124 I tried to solve Laplace equation using mainly information from Pierre Grivet's
125 book (Electron Optics) - [1].

126 There are few editions of this book (1965, 1972). Second edition (1972) con-
127 tains explanation of relaxation method (page 38).

128 More generalized approaches has been drafted by James R. Nagel - [2].
129 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).

130
131 There are also publications edited by Albert Septier: Focusing of Charged
132 Particles [3] and Applied Charged Particle Optics (part A). [4].

133 I have also found some ideas in publication of D W O Heddle: Electrostatic
134 Lens Systems [5] (especially using PC computers to solve electrostatic prob-
135 lems).

136 I have also found (brief) description of by - hand solving of Laplace equa-
137 tion by Bohdan Paszkowski - [6] (Polish edition). English translation of this book
138 also exists - [7].

139
140 I would like to thank many people, who helped me with this challenge. Espe-
141 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),
142 who enabled me to use SIMION and MATLAB software while writing master's
143 thesis about electron optical systems at University of Maria Curie - Skłodowska
144 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-
145 sion about numerical methods. What is more, my colleague Bartosz in 2012
146 had explained me general problems with software efficiency. So he had also
147 contributed significantly to the idea of Liebmann software (especially using C
148 language).

149 **4 Explanation of symbols in calculations**

- 150 • P_i - i -th mesh node
- 151 • V_i - value of electrostatic potential at node P_i . Unit - [V]
- 152 • h - mesh step (for example h_x - mesh step in x direction). Unit - [mm]
- 153 • $g_{i+/-}$ - gradient in direction i (for example $g_{1x-} = \frac{V_1 - V_{1x-}}{h_x}$. Unit - $[\frac{V}{mm}]$)
- 154 • i_{row} - index of row in mesh. Values of $i_{row} = 1, 2, \dots, size_row$
- 155 • i_{col} - index of column in mesh. Values of $i_{col} = 1, 2, \dots, size_col$

156 Symbols in final relaxation formulae

157 xyLV_RELAX5_P1_A

- 158 • xy - coordinates (2D, planar)

- 159 • LV - Laplace equation in vacuum (no dielectrics)
- 160 • RELAX5 - 5- point relaxation method
- 161 • P1 - relaxation scheme for point P1 (in general P1 .. P9)
- 162 • A - mesh type A (in general A .. D)

163 **5 Mesh XY - type A**

164 $h_x \neq h_y$

165 gradient V outside a mesh exists

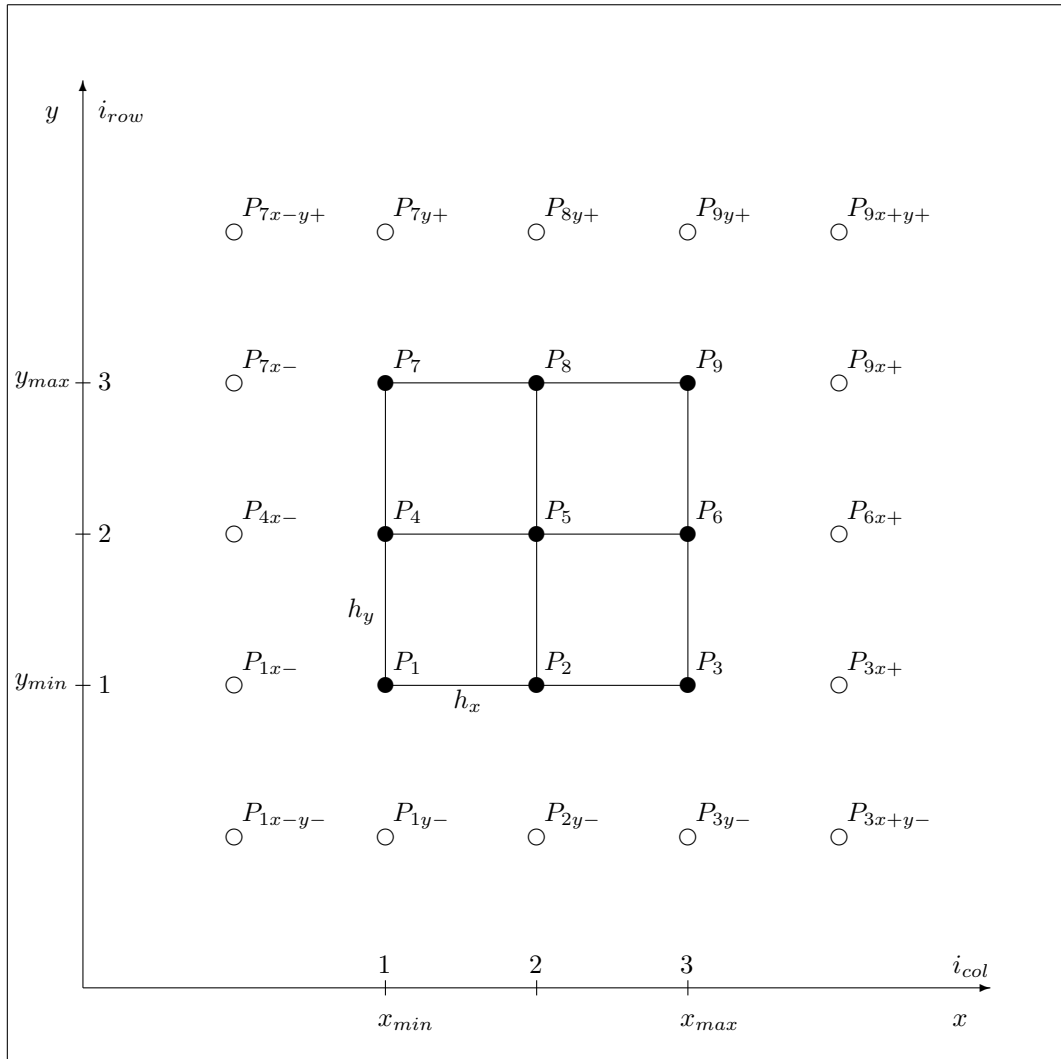


Figure 1: Mesh XY type A

166 **6 Mesh XY - type B**

167 $h_x \neq h_y$

168 gradient V outside a mesh does not exist

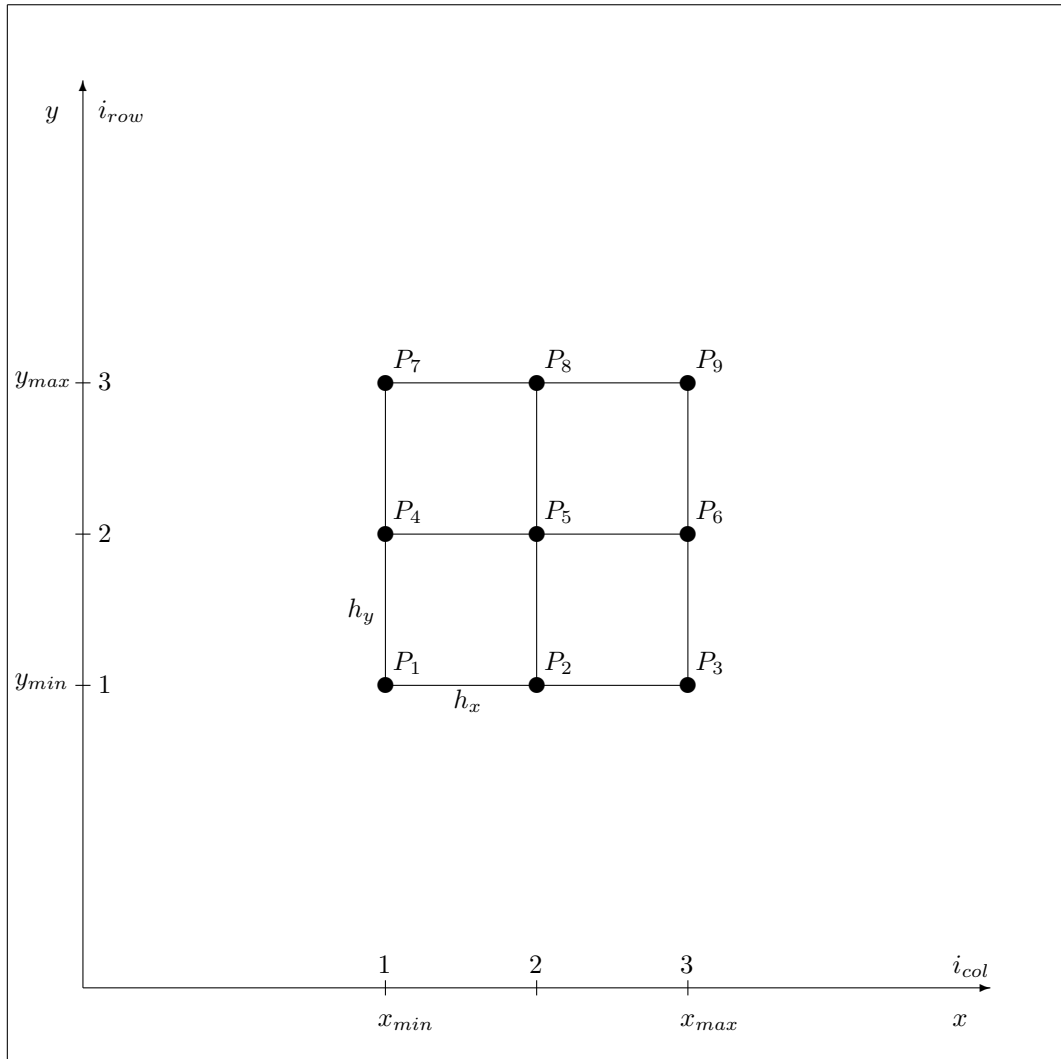


Figure 2: Mesh XY type B

169 **7 Mesh XY - type C**

170 $h_x = h_y = h$

171 gradient V outside a mesh exists

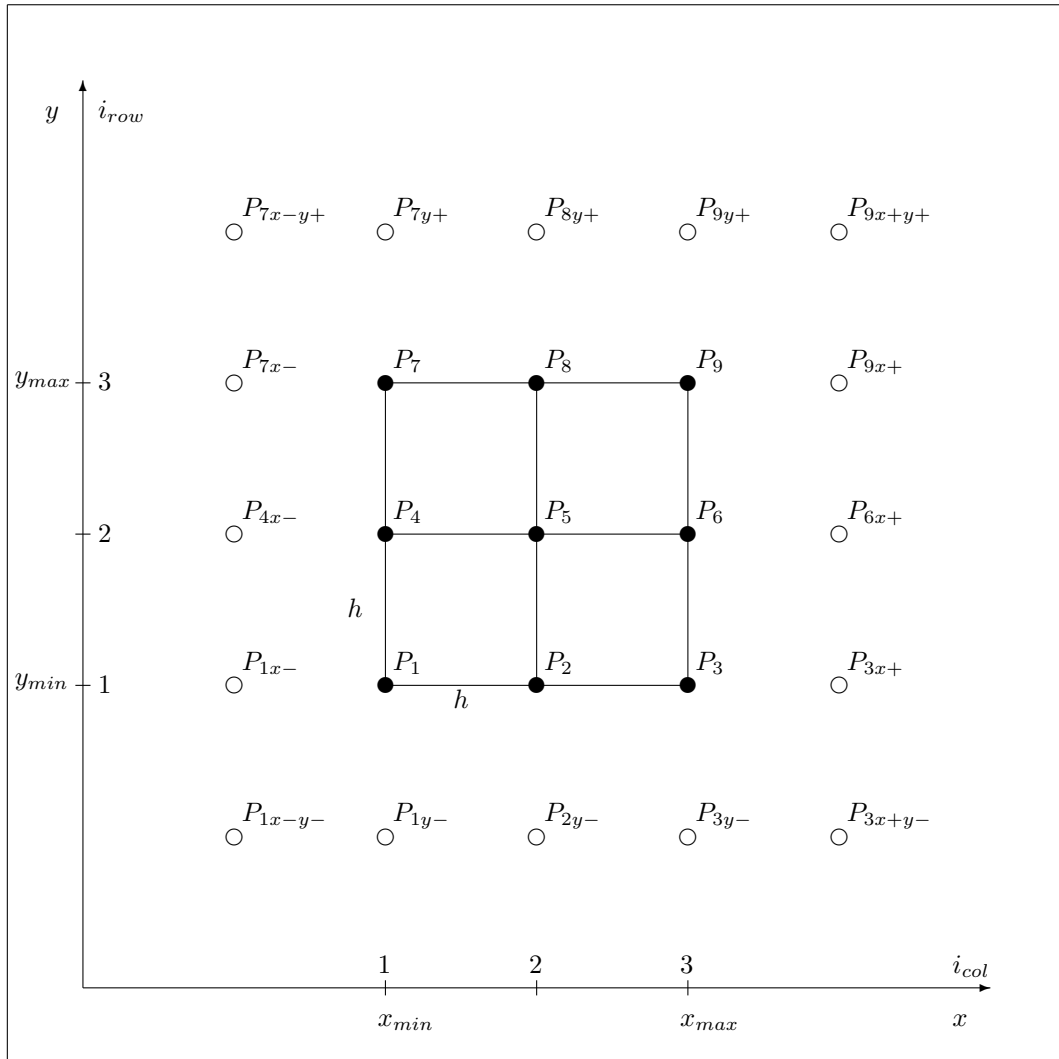


Figure 3: Mesh XY type C

172 **8 Mesh XY - type D**

173 $h_x = h_y = h$

174 gradient V outside a mesh does not exist

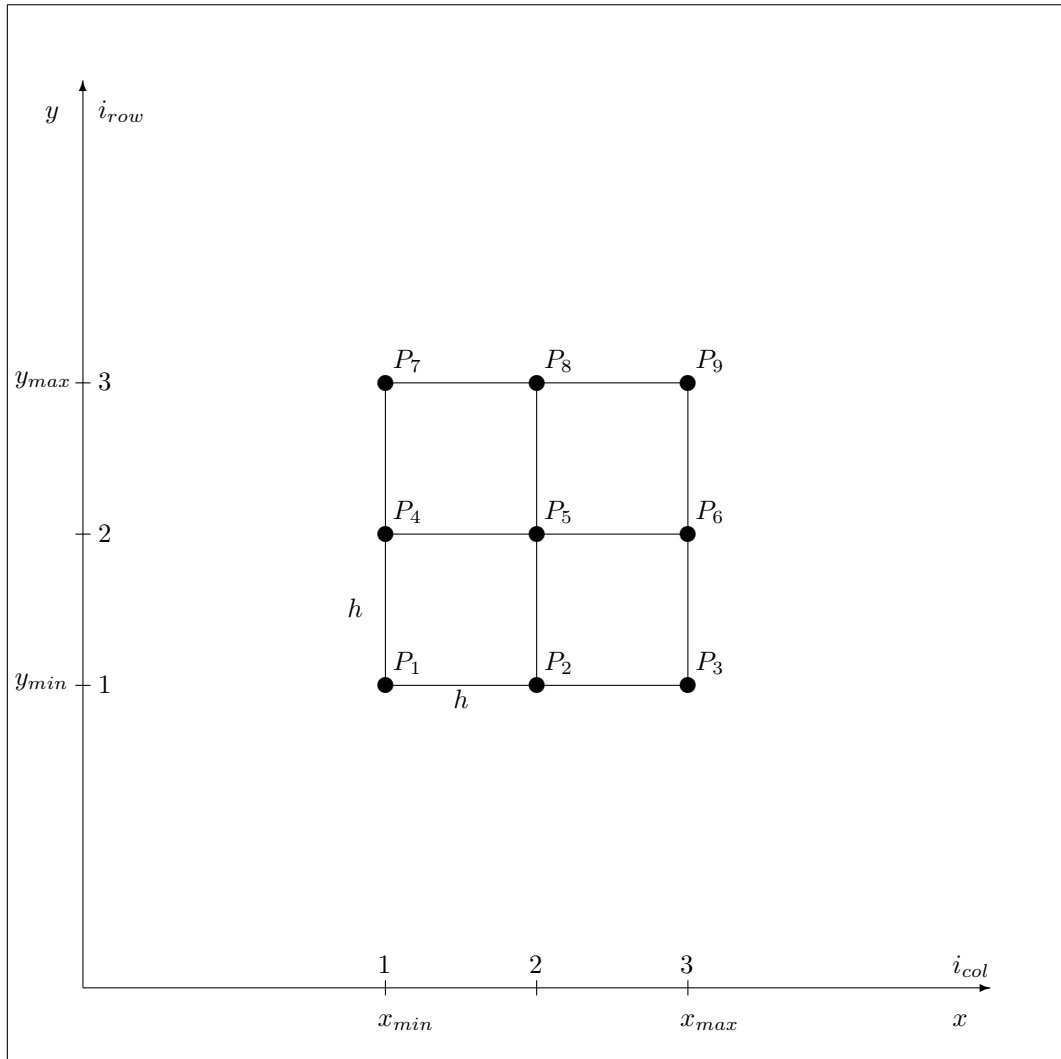


Figure 4: Mesh XY type D

175 **9 Example of A-type mesh in ANSI C**

176 Example of A- type mesh in ANSI C program. The mesh is represented by 2
 177 dimensional array of double precision numbers. Rows and columns in mesh
 178 are numbered from 1 (this was my choice) instead of default 0 (as usual in C
 179 language). This choice has pros and cons. It is easier to calculate mesh size
 180 (size_row * size_col). Access to each node can be also more intuitive, but logic
 181 in each library function must contain this shift between node ordering styles.

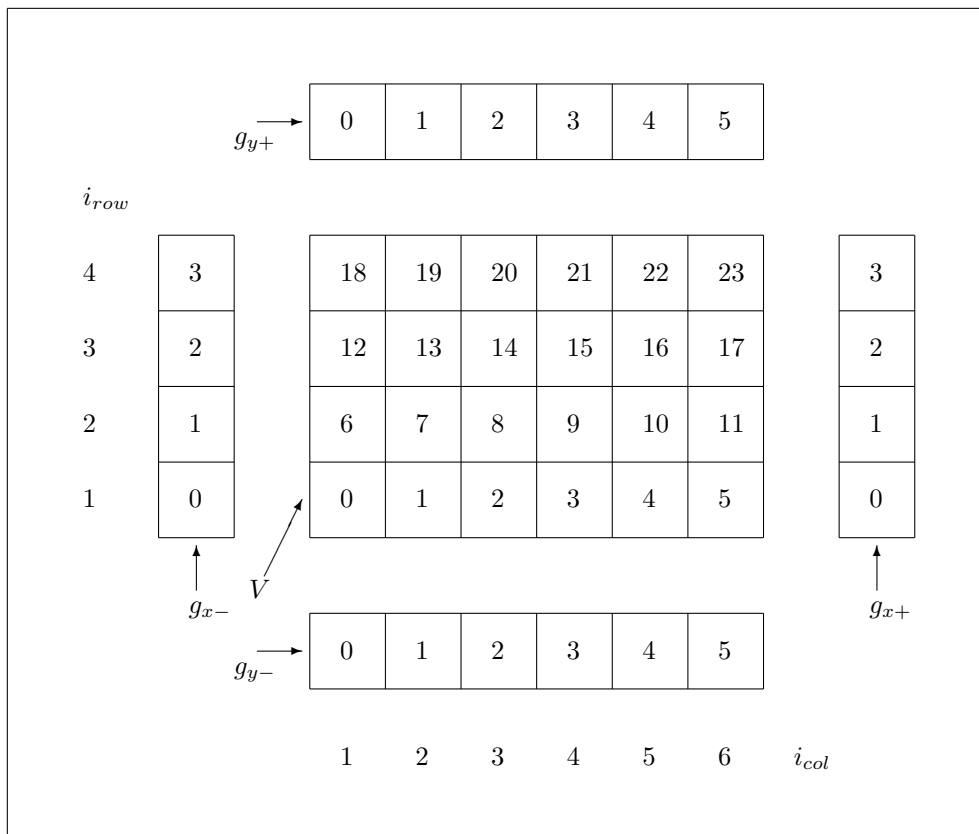


Figure 5: ANSI C - mesh XY type A

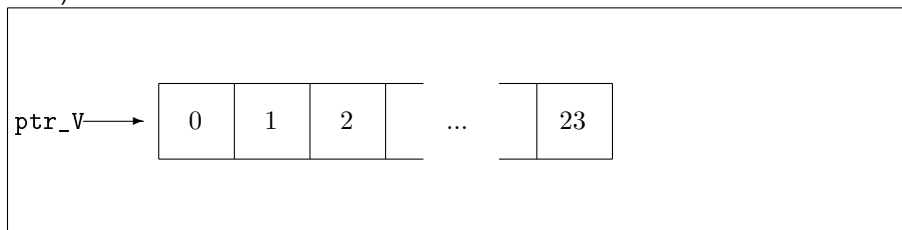
- 182 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 183 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 184 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 185 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 186 • $V \equiv \text{double* ptr_V}$
- 187 • `unsigned int size_row == 4`

```

188     • unsigned int size_col == 6
189     • unsigned int i_row == 1, 2, .., 4
190     • unsigned int i_col == 1,2, .., 6
191     • double h_x == 1.0 [mm]
192     • double h_y == 2.0 [mm]

```

193 The following picture describes analogous version of ptr_V mesh, which
194 can be dynamically allocated on heap by pointer method. The mesh is rep-
195 resented by single block of memory. The numbers of rows and columns are
196 also known, so each node can be also accessed by appropriate index (memory
197 address).



198
199 Each mesh point has its unique index (let's say icp - (index of central
200 point)), which can be determined, if we know indices of row and column (i_row,
201 i_col).

$$icp == (i_row - 1) * size_col + i_col - 1 \quad (9.1)$$

202 For example for each point of a mesh indices of row and column have val-
203 ues:

$$\begin{aligned}
i_row &== 1, 2, \dots, size_row \\
i_col &== 1, 2, \dots, size_col
\end{aligned} \quad (9.2)$$

204 **10 Example of B-type mesh in ANSI C**

205 Example of B- type mesh in ANSI C program. The mesh is analogous to A -
206 type mesh. There are no electric field gradients on mesh borders.

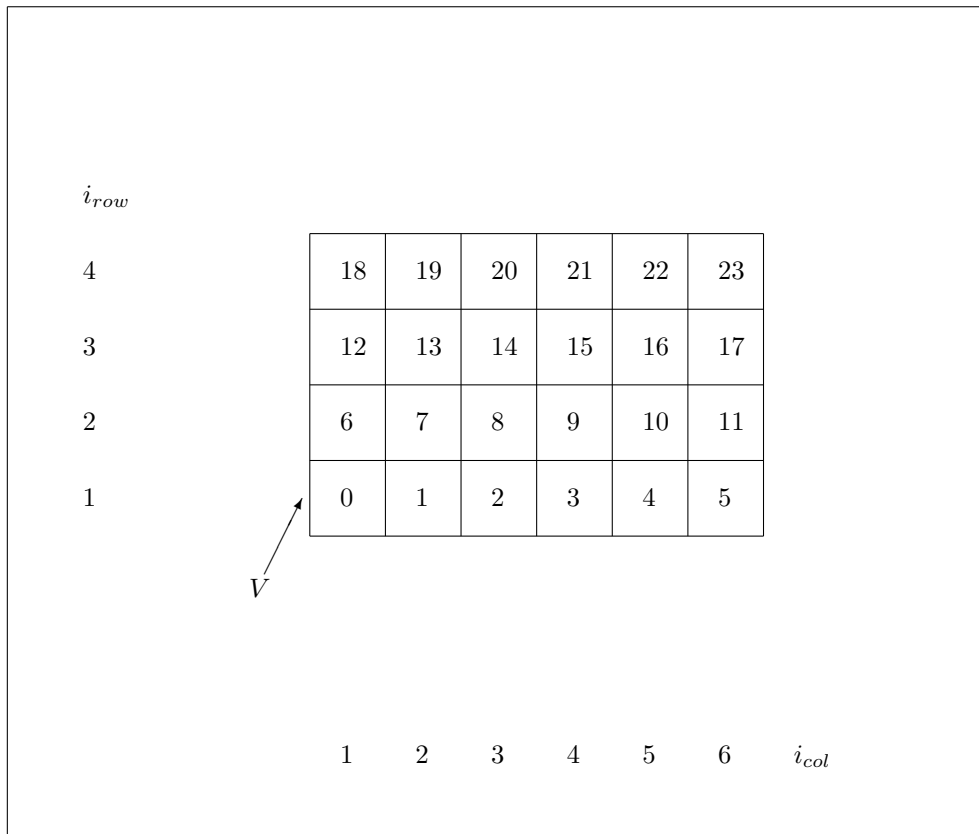


Figure 6: ANSI C - mesh XY type B

- 207 • $V \equiv \text{double* ptr}_V$
- 208 • `unsigned int size_row == 4`
- 209 • `unsigned int size_col == 6`
- 210 • `unsigned int i_row == 1, 2, .., 4`
- 211 • `unsigned int i_col == 1,2, .., 6`
- 212 • `double h_x == 1.0 [mm]`
- 213 • `double h_y == 2.0 [mm]`

214 **11 Example of C-type mesh in ANSI C**

215 Example of C- type mesh in ANSI C program. The mesh is analogous to A -
 216 type mesh. Just mesh mesh step $h_x = h_y = h$.

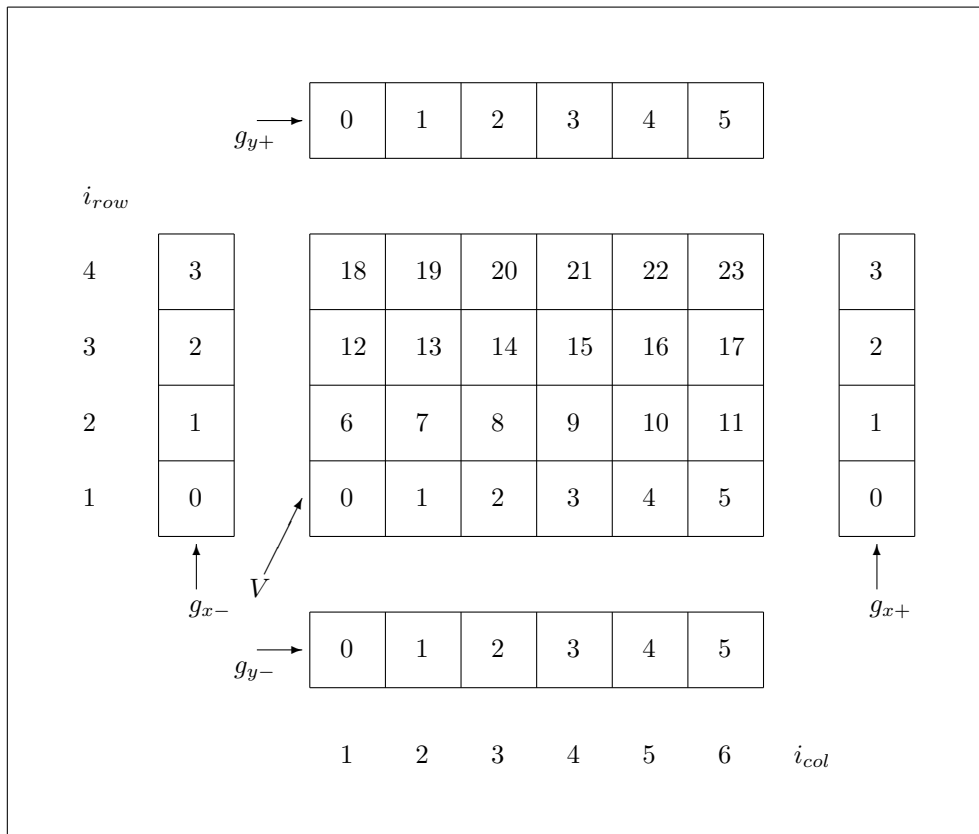


Figure 7: ANSI C - mesh XY type C

- 217 • $g_{x-} \equiv \text{double* ptr_gX_minus}$
- 218 • $g_{x+} \equiv \text{double* ptr_gX_plus}$
- 219 • $g_{y-} \equiv \text{double* ptr_gY_minus}$
- 220 • $g_{y+} \equiv \text{double* ptr_gY_plus}$
- 221 • $V \equiv \text{double* ptr_V}$
- 222 • `unsigned int size_row == 4`
- 223 • `unsigned int size_col == 6`
- 224 • `unsigned int i_row == 1, 2, .., 4`

```
225     • unsigned int i_col == 1,2, .., 6
226     • double h == 1.0 [mm]
```


227 **12 Example of D-type mesh in ANSI C**

228 Example of D- type mesh in ANSI C program. The mesh is analogous to B -
229 type mesh. Just $h_x = h_y = h$.

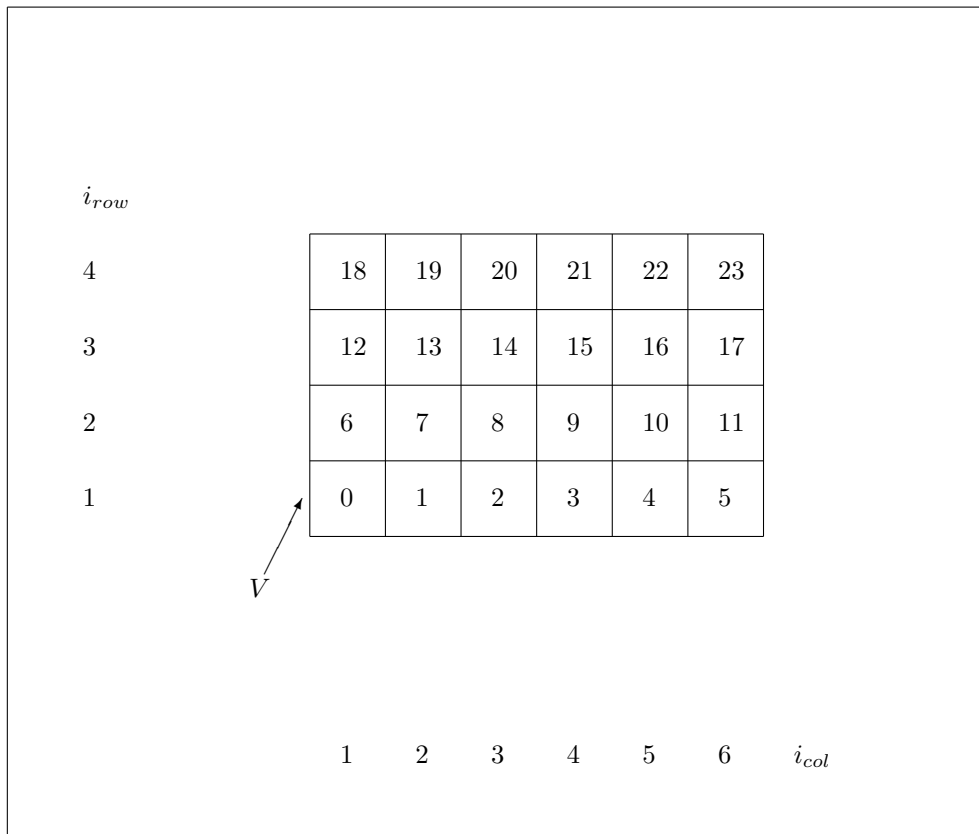


Figure 8: ANSI C - mesh XY type D

- 230 • $V \equiv \text{double* ptr}_V$
- 231 • `unsigned int size_row == 4`
- 232 • `unsigned int size_col == 6`
- 233 • `unsigned int i_row == 1, 2, ..., 4`
- 234 • `unsigned int i_col == 1,2, ..., 6`
- 235 • `double h == 1.0 [mm]`

236 **13 Relaxation formula for node P1**

237 **13.1 Node description**

238 Left, bottom corner of mesh XY.

239 **13.2 Calculation of relaxation formula**

240 Laplace equation at node P_1

$$\nabla^2 (V_{(x,y)})_{P_1} = 0 \quad (13.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} = 0 \quad (13.2)$$

241 Approximation of partial derivatives of $V_{(x,y)}$ at node P_1

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_x} - \frac{V_1 - V_{1x-}}{h_x}}{h_x} = \frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} \quad (13.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_y} - \frac{V_1 - V_{1y-}}{h_y}}{h_y} = \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} \quad (13.4)$$

242 Let us substitute approximations to Laplace equation.

$$\frac{V_2 - V_1}{h_x^2} - \frac{g_{1x-}}{h_x} + \frac{V_4 - V_1}{h_y^2} - \frac{g_{1y-}}{h_y} = 0 \quad (13.5)$$

243 Let us find V_1

$$V_1 = ? \quad (13.6)$$

$$\frac{V_2 - V_1}{h_x^2} + \frac{V_4 - V_1}{h_y^2} = \frac{g_{1x-}}{h_x} + \frac{g_{1y-}}{h_y} \quad (13.7)$$

244 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (13.8)$$

245 We obtain

$$V_2 h_y^2 - V_1 h_y^2 + V_4 h_x^2 - V_1 h_x^2 = g_{1x-} h_x h_y^2 + g_{1y-} h_x^2 h_y \quad (13.9)$$

$$V_1 (h_x^2 + h_y^2) = V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y \quad (13.10)$$

246 **13.3 Final forms of relaxation formula**

247 **13.3.1 xyLV_RELAX5_P1_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} \neq 0 \\ 248 \quad V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2 - g_{1x-} h_x h_y^2 - g_{1y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (13.11)$$

249 **13.3.2 xyLV_RELAX5_P1_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 h_y^2 + V_4 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (13.12)$$

250 **13.3.3 xyLV_RELAX5_P1_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} \neq 0 \\ V_1 &= \frac{V_2 + V_4 - g_{1x-} h - g_{1y-} h}{2} \end{aligned} \quad (13.13)$$

251 **13.3.4 xyLV_RELAX5_P1_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{1x-}, g_{1y-} = 0 \\ V_1 &= \frac{V_2 + V_4}{2} \end{aligned} \quad (13.14)$$

252 **14 Relaxation formula for node P2**

253 **14.1 Node description**

254 Bottom edge of mesh XY.

255 **14.2 Calculation of relaxation formula**

256 Laplace equation at node P_2

$$\nabla^2 (V_{(x,y)})_{P_2} = 0 \quad (14.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} = 0 \quad (14.2)$$

257 Approximation of partial derivatives of $V_{(x,y)}$ at node P_2

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_x} - \frac{V_2 - V_1}{h_x}}{h_x} = \frac{V_1 + V_3 - 2V_2}{h_x^2} \quad (14.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_y} - \frac{V_2 - V_{2y-}}{h_y}}{h_y} = \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} \quad (14.4)$$

258 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} - \frac{g_{2y-}}{h_y} = 0 \quad (14.5)$$

259 Let us find V_2

$$V_2 = ? \quad (14.6)$$

$$\frac{V_1 + V_3 - 2V_2}{h_x^2} + \frac{V_5 - V_2}{h_y^2} = \frac{g_{2y-}}{h_y} \quad (14.7)$$

260 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (14.8)$$

261 We obtain

$$V_1 h_y^2 + V_3 h_y^2 - 2V_2 h_y^2 + V_5 h_x^2 = g_{2y-} h_x^2 h_y \quad (14.9)$$

$$V_2 (h_x^2 + h_y^2) = (V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y \quad (14.10)$$

262 **14.3 Final forms of relaxation formula**

263 **14.3.1 xyLV_RELAX5_P2_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2 - g_{2y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (14.11)$$

264 **14.3.2 xyLV_RELAX5_P2_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{2y-} &= 0 \\ V_2 &= \frac{(V_1 + V_3) h_y^2 + V_5 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (14.12)$$

265 **14.3.3 xyLV_RELAX5_P2_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &\neq 0 \\ V_2 &= \frac{V_1 + V_3 + V_5 - g_{2y-} h}{3} \end{aligned} \quad (14.13)$$

266 **14.3.4 xyLV_RELAX5_P2_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{2y-} &= 0 \\ V_2 &= \frac{V_1 + V_3 + V_5}{3} \end{aligned} \quad (14.14)$$

267 **15 Relaxation formula for node P3**

268 **15.1 Node description**

269 Right, bottom corner of mesh XY.

270 **15.2 Calculation of relaxation formula**

271 Laplace equation at node P_3

$$\nabla^2 (V_{(x,y)})_{P_3} = 0 \quad (15.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} = 0 \quad (15.2)$$

272 Approximation of partial derivatives of $V_{(x,y)}$ at node P_3

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_3} \approx \frac{V_{3x+} - V_3 - \frac{V_3 - V_2}{h_x}}{h_x} = \frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} \quad (15.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_y} - \frac{V_3 - V_{3y-}}{h_y}}{h_y} = \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} \quad (15.4)$$

273 Let us substitute approximations to Laplace equation.

$$\frac{g_{3x+}}{h_x} + \frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} - \frac{g_{3y-}}{h_y} = 0 \quad (15.5)$$

274 Let us find V_3

$$V_3 = ? \quad (15.6)$$

$$\frac{V_2 - V_3}{h_x^2} + \frac{V_6 - V_3}{h_y^2} = \frac{g_{3y-}}{h_y} - \frac{g_{3x+}}{h_x} \quad (15.7)$$

275 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (15.8)$$

276 We obtain

$$V_2 h_y^2 - V_3 h_y^2 + V_6 h_x^2 - V_3 h_x^2 = g_{3y-} h_x^2 h_y - g_{3x+} h_x h_y^2 \quad (15.9)$$

$$V_3 (h_x^2 + h_y^2) = V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y \quad (15.10)$$

277 **15.3 Final forms of relaxation formula**

278 **15.3.1 xyLV_RELAX5_P3_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2 + g_{3x+} h_x h_y^2 - g_{3y-} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (15.11)$$

279 **15.3.2 xyLV_RELAX5_P3_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 h_y^2 + V_6 h_x^2}{h_x^2 + h_y^2} \end{aligned} \quad (15.12)$$

280 **15.3.3 xyLV_RELAX5_P3_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} \neq 0 \\ V_3 &= \frac{V_2 + V_6 + g_{3x+} h - g_{3y-} h}{2} \end{aligned} \quad (15.13)$$

281 **15.3.4 xyLV_RELAX5_P3_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{3x+}, g_{3y-} = 0 \\ V_3 &= \frac{V_2 + V_6}{2} \end{aligned} \quad (15.14)$$

282 **16 Relaxation formula for node P4**

283 **16.1 Node description**

284 Left edge of mesh XY.

285 **16.2 Calculation of relaxation formula**

286 Laplace equation at node P_4

$$\nabla^2 (V_{(x,y)})_{P_4} = 0 \quad (16.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} = 0 \quad (16.2)$$

287 Approximation of partial derivatives of $V_{(x,y)}$ at node P_4

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_4}{h_x} - \frac{V_4 - V_{4x-}}{h_x}}{h_x} = \frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} \quad (16.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_y} - \frac{V_4 - V_1}{h_y}}{h_y} = \frac{V_1 + V_7 - 2V_4}{h_y^2} \quad (16.4)$$

288 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_4}{h_x^2} - \frac{g_{4x-}}{h_x} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = 0 \quad (16.5)$$

289 Let us find V_4

$$V_4 = ? \quad (16.6)$$

$$\frac{V_5 - V_4}{h_x^2} + \frac{V_1 + V_7 - 2V_4}{h_y^2} = \frac{g_{4x-}}{h_x} \quad (16.7)$$

290 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (16.8)$$

291 We obtain

$$V_5 h_y^2 - V_4 h_y^2 + V_1 h_x^2 + V_7 h_x^2 - 2V_4 h_x^2 = g_{4x-} h_x h_y^2 \quad (16.9)$$

$$V_4 (2h_x^2 + h_y^2) = (V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2 \quad (16.10)$$

292 **16.3 Final forms of relaxation formula**

293 **16.3.1 xyLV_RELAX5_P4_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2 - g_{4x-} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.11)$$

294 **16.3.2 xyLV_RELAX5_P4_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{4x-} &= 0 \\ V_2 &= \frac{(V_1 + V_7) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (16.12)$$

295 **16.3.3 xyLV_RELAX5_P4_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &\neq 0 \\ V_4 &= \frac{V_1 + V_5 + V_7 - g_{4x-} h}{3} \end{aligned} \quad (16.13)$$

296 **16.3.4 xyLV_RELAX5_P4_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{4x-} &= 0 \\ V_4 &= \frac{V_1 + V_5 + V_7}{3} \end{aligned} \quad (16.14)$$

297 **17 Relaxation formula for node P5**

298 **17.1 Node description**

299 Node inside a mesh XY.

300 **17.2 Calculation of relaxation formula**

301 Laplace equation at node P_5

$$\nabla^2 (V_{(x,y)})_{P_5} = 0 \quad (17.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} = 0 \quad (17.2)$$

302 Approximation of partial derivatives of $V_{(x,y)}$ at node P_5

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_x} - \frac{V_5 - V_4}{h_x}}{h_x} = \frac{V_4 + V_6 - 2V_5}{h_x^2} \quad (17.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_y} - \frac{V_5 - V_2}{h_y}}{h_y} = \frac{V_2 + V_8 - 2V_5}{h_y^2} \quad (17.4)$$

303 Let us substitute approximations to Laplace equation.

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.5)$$

304 Let us find V_5

$$V_5 = ? \quad (17.6)$$

$$\frac{V_4 + V_6 - 2V_5}{h_x^2} + \frac{V_2 + V_8 - 2V_5}{h_y^2} = 0 \quad (17.7)$$

305 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (17.8)$$

306 We obtain

$$V_4 h_y^2 + V_6 h_y^2 - 2V_5 h_y^2 + V_2 h_x^2 + V_8 h_x^2 - 2V_5 h_x^2 = 0 \quad (17.9)$$

$$2V_5 (h_x^2 + h_y^2) = (V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2 \quad (17.10)$$

307 **17.3 Final forms of relaxation formula**

308 **17.3.1 xyLV_RELAX5_P5_A**

$$h_x \neq h_y$$

309 No gradients g inside mesh are considered.

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.11)$$

310 **17.3.2 xyLV_RELAX5_P5_B**

$$h_x \neq h_y$$

311 Relaxation formula is the same as xyLV_RELAX5_P5_A

$$V_5 = \frac{(V_2 + V_8) h_x^2 + (V_4 + V_6) h_y^2}{2(h_x^2 + h_y^2)} \quad (17.12)$$

312 **17.3.3 xyLV_RELAX5_P5_C**

$$h_x = h_y = h$$

313 No gradients g inside mesh are considered.

314 The formula simplifies, so no g and h terms are necessary.

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.13)$$

315 **17.3.4 xyLV_RELAX5_P5_D**

$$h_x = h_y = h$$

316 The formula also simplifies.

317

318 Relaxation formula is the same as xyLV_RELAX5_P5_C

$$V_5 = \frac{V_2 + V_4 + V_6 + V_8}{4} \quad (17.14)$$

319 **18 Relaxation formula for node P6**

320 **18.1 Node description**

321 Right edge of mesh XY.

322 **18.2 Calculation of relaxation formula**

323 Laplace equation at node P_6

$$\nabla^2 (V_{(x,y)})_{P_6} = 0 \quad (18.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} = 0 \quad (18.2)$$

324 Approximation of partial derivatives of $V_{(x,y)}$ at node P_6

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_6} \approx \frac{V_{6x+} - V_6}{h_x} - \frac{V_6 - V_5}{h_x} = \frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} \quad (18.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_6} \approx \frac{V_9 - V_6}{h_y} - \frac{V_6 - V_3}{h_y} = \frac{V_3 + V_9 - 2V_6}{h_y^2} \quad (18.4)$$

325 Let us substitute approximations to Laplace equation.

$$\frac{g_{6x+}}{h_x} + \frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = 0 \quad (18.5)$$

326 Let us find V_6

$$V_6 = ? \quad (18.6)$$

$$\frac{V_5 - V_6}{h_x^2} + \frac{V_3 + V_9 - 2V_6}{h_y^2} = -\frac{g_{6x+}}{h_x} \quad (18.7)$$

327 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (18.8)$$

328 We obtain

$$V_5 h_y^2 - V_6 h_y^2 + V_3 h_x^2 + V_9 h_x^2 - 2V_6 h_x^2 = -g_{6x+} h_x h_y^2 \quad (18.9)$$

$$V_6 (2h_x^2 + h_y^2) = (V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2 \quad (18.10)$$

329 **18.3 Final forms of relaxation formula**

330 **18.3.1 xyLV_RELAX5_P6_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2 + g_{6x+} h_x h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.11)$$

331 **18.3.2 xyLV_RELAX5_P6_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{6x+} &= 0 \\ V_6 &= \frac{(V_3 + V_9) h_x^2 + V_5 h_y^2}{2h_x^2 + h_y^2} \end{aligned} \quad (18.12)$$

332 **18.3.3 xyLV_RELAX5_P6_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &\neq 0 \\ V_6 &= \frac{V_3 + V_5 + V_9 + g_{6x+} h}{3} \end{aligned} \quad (18.13)$$

333 **18.3.4 xyLV_RELAX5_P6_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{6x+} &= 0 \\ V_6 &= \frac{V_3 + V_5 + V_9}{3} \end{aligned} \quad (18.14)$$

334 **19 Relaxation formula for node P7**

335 **19.1 Node description**

336 Left, upper corner of mesh XY.

337 **19.2 Calculation of relaxation formula**

338 Laplace equation at node P_7

$$\nabla^2 (V_{(x,y)})_{P_7} = 0 \quad (19.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} = 0 \quad (19.2)$$

339 Approximation of partial derivatives of $V_{(x,y)}$ at node P_7

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_x} - \frac{V_7 - V_{7x-}}{h_x}}{h_x} = \frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} \quad (19.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_7} \approx \frac{\frac{V_{7y+} - V_7}{h_y} - \frac{V_7 - V_4}{h_y}}{h_y} = \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} \quad (19.4)$$

340 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_7}{h_x^2} - \frac{g_{7x-}}{h_x} + \frac{V_4 - V_7}{h_y^2} + \frac{g_{7y+}}{h_y} = 0 \quad (19.5)$$

341 Let us find V_7

$$V_7 = ? \quad (19.6)$$

$$\frac{V_8 - V_7}{h_x^2} + \frac{V_4 - V_7}{h_y^2} = \frac{g_{7x-}}{h_x} - \frac{g_{7y+}}{h_y} \quad (19.7)$$

342 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (19.8)$$

343 We obtain

$$V_8 h_y^2 - V_7 h_y^2 + V_4 h_x^2 - V_7 h_x^2 = g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.9)$$

$$V_7 (h_x^2 + h_y^2) = V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 - g_{7y+} h_x^2 h_y \quad (19.10)$$

344 **19.3 Final forms of relaxation formula**

345 **19.3.1 xyLV_RELAX5_P7_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2 - g_{7x-} h_x h_y^2 + g_{7y+} h_x^2 h_y}{(h_x^2 + h_y^2)} \end{aligned} \quad (19.11)$$

346 **19.3.2 xyLV_RELAX5_P7_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (19.12)$$

347 **19.3.3 xyLV_RELAX5_P7_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &\neq 0 \\ V_7 &= \frac{V_4 + V_8 - g_{7x-} h + g_{7y+} h}{2} \end{aligned} \quad (19.13)$$

348 **19.3.4 xyLV_RELAX5_P7_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{7x-}, g_{7y+} &= 0 \\ V_7 &= \frac{V_4 + V_8}{2} \end{aligned} \quad (19.14)$$

349 **20 Relaxation formula for node P8**

350 **20.1 Node description**

351 Upper edge of mesh XY.

352 **20.2 Calculation of relaxation formula**

353 Laplace equation at node P_8

$$\nabla^2 (V_{(x,y)})_{P_8} = 0 \quad (20.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} = 0 \quad (20.2)$$

354 Approximation of partial derivatives of $V_{(x,y)}$ at node P_8

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_x} - \frac{V_8 - V_7}{h_x}}{h_x} = \frac{V_7 + V_9 - 2V_8}{h_x^2} \quad (20.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_8} \approx \frac{\frac{V_{8y+} - V_8}{h_y} - \frac{V_8 - V_5}{h_y}}{h_y} = \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} \quad (20.4)$$

355 Let us substitute approximations to Laplace equation.

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} + \frac{g_{8y+}}{h_y} = 0 \quad (20.5)$$

356 Let us find V_8

$$V_8 = ? \quad (20.6)$$

$$\frac{V_7 + V_9 - 2V_8}{h_x^2} + \frac{V_5 - V_8}{h_y^2} = -\frac{g_{8y+}}{h_y} \quad (20.7)$$

357 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (20.8)$$

358 We obtain

$$V_7 h_y^2 + V_9 h_y^2 - 2V_8 h_y^2 + V_5 h_x^2 - V_8 h_x^2 = -g_{8y+} h_x^2 h_y \quad (20.9)$$

$$V_8 (h_x^2 + 2h_y^2) = (V_7 + V_9) h_y^2 + V_5 h_x^2 + g_{8y+} h_x^2 h_y \quad (20.10)$$

359 **20.3 Final forms of relaxation formula**

360 **20.3.1 xyLV_RELAX5_P8_A**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2 + g_{8y+} h_x^2 h_y}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.11)$$

361 **20.3.2 xyLV_RELAX5_P8_B**

$$\begin{aligned} h_x &\neq h_y \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 h_x^2 + (V_7 + V_9) h_y^2}{h_x^2 + 2h_y^2} \end{aligned} \quad (20.12)$$

362 **20.3.3 xyLV_RELAX5_P8_C**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &\neq 0 \\ V_8 &= \frac{V_5 + V_7 + V_9 + g_{8y+} h}{3} \end{aligned} \quad (20.13)$$

363 **20.3.4 xyLV_RELAX5_P8_D**

$$\begin{aligned} h_x &= h_y = h \\ g_{8y+} &= 0 \\ V_8 &= \frac{V_5 + V_7 + V_9}{3} \end{aligned} \quad (20.14)$$

364 **21 Relaxation formula for node P9**

365 **21.1 Node description**

366 Right, upper corner of mesh XY.

367 **21.2 Calculation of relaxation formula**

368 Laplace equation at node P_9

$$\nabla^2 (V_{(x,y)})_{P_9} = 0 \quad (21.1)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} + \left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} = 0 \quad (21.2)$$

369 Approximation of partial derivatives of $V_{(x,y)}$ at node P_9

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial x^2} \right)_{P_9} \approx \frac{\frac{V_{9x+} - V_9}{h_x} - \frac{V_9 - V_8}{h_x}}{h_x} = \frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} \quad (21.3)$$

$$\left(\frac{\partial^2 V_{(x,y)}}{\partial y^2} \right)_{P_9} \approx \frac{\frac{V_{9y+} - V_9}{h_y} - \frac{V_9 - V_6}{h_y}}{h_y} = \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} \quad (21.4)$$

370 Let us substitute approximations to Laplace equation.

$$\frac{V_8 - V_9}{h_x^2} + \frac{g_{9x+}}{h_x} + \frac{V_6 - V_9}{h_y^2} + \frac{g_{9y+}}{h_y} = 0 \quad (21.5)$$

371 Let us find V_9

$$V_9 = ? \quad (21.6)$$

$$\frac{V_8 - V_9}{h_x^2} + \frac{V_8 - V_9}{h_x^2} = -\frac{g_{9x+}}{h_x} - \frac{g_{9y+}}{h_y} \quad (21.7)$$

372 Let us multiply both sides

$$| \cdot h_x^2 h_y^2 \quad (21.8)$$

373 We obtain

$$V_8 h_y^2 - V_9 h_y^2 + V_6 h_x^2 - V_9 h_x^2 = -g_{9x+} h_x h_y^2 - g_{9y+} h_x^2 h_y \quad (21.9)$$

$$V_9 (h_x^2 + h_y^2) = V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y \quad (21.10)$$

374 **21.3 Final forms of relaxation formula**

375 **21.3.1 xyLV_RELAX5_P9_A**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2 + g_{9x+} h_x h_y^2 + g_{9y+} h_x^2 h_y}{h_x^2 + h_y^2} \end{aligned} \quad (21.11)$$

376 **21.3.2 xyLV_RELAX5_P9_B**

$$\begin{aligned} & h_x \neq h_y \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 h_x^2 + V_8 h_y^2}{h_x^2 + h_y^2} \end{aligned} \quad (21.12)$$

377 **21.3.3 xyLV_RELAX5_P9_C**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} \neq 0 \\ V_9 &= \frac{V_6 + V_8 + g_{9x+} h + g_{9y+} h}{2} \end{aligned} \quad (21.13)$$

378 **21.3.4 xyLV_RELAX5_P9_D**

$$\begin{aligned} & h_x = h_y = h \\ & g_{9x+}, g_{9y+} = 0 \\ V_9 &= \frac{V_6 + V_8}{2} \end{aligned} \quad (21.14)$$

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